Capillary Force Actuators

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Dean, School of Engineering and Applied Science

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Abstract

Existing micro actuation technologies -electrostatic, electromagnetic, piezoelectric, and thermal- have significant limitations that impede their use in microdevices. Many applications demand greater force, larger stroke, lower power, or higher bandwidth than is possible with these technologies. Furthermore, some of these actuators are very difficult to integrate into microfabrication. This dissertation examines a novel micro actuation technology, Capillary Force Actuation (CFA), which takes advantage of microscale capillary bridges to produce 10 to 100 times greater force than the most commonly used micro actuators. A standard configuration for capillary force actuators is introduced and the electromechanical principles underlying the actuation principle are studied. Semi-analytical investigation of the capillary bridge profile with and without applied electric field is performed. Parameters characterizing the device configuration are introduced and design charts in terms of these parameters are developed. Design optimization and limits are studied using an analytical approximation of the actuator force produced. It is shown that if the actuation voltage is fixed, the maximum force achievable is independent of the surface tension of the liquid. It is further argued that for given material properties there exist an optimal thickness of dielectric for which the maximum force may be achieved before dielectric breakdown or contact angle saturation is observed. Numerical methods for calculating the force are introduced for more complex configurations. The stability of the capillary bridge is also investigated and the implications this has for actuator design are examined. It is shown that for typical values of configuration geometry, the capillary bridge will remain stable during actuation. Alternative configurations of CFA are considered. The boundary condition problem
governing the bridge shape for each of these cases is derived and solution methods are presented. Examples of the various alternative configurations are provided and a comparison of the force level to that of the standard configuration is made. It is shown that the standard configuration is more effective than configurations with pinned contact line or constant contact angle on one surface for maximum change in force. The rate of the force change however may be increased by employing a surface energy gradient on the surfaces. For a device with only one active surface, it is shown that pinning the passive side would be most effective for maximum change in force.
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Nomenclature

**Arabic**

- $a$ Aspect ratio
- $a^*$ Aspect ratio for the pinned case
- $C$ Capacitance of the gap between electrodes in electrostatic actuator
- $C_d$ Capacitance of dielectric
- $c$ Evaporation constant
- $D$ Diameter of the pinned boundary condition
- $d$ Maximum deflection of membrane
- $d\psi_1$ Angle corresponding to a small section of surface on one of the planes perpendicular to the tangential plane
- $d\psi_2$ Angle corresponding to a small section of surface on plane perpendicular to the tangential plane and the first plane chosen
- $E$ Electric field inside dielectric
- $E_{mem}$ Young’s modulus
- $\bar{E}$ Electric field
- $E_n$ Component of electric field normal to the liquid surface
- $e_b$ Breakdown field strength
- $F_b$ Approximate maximum force achievable considering breakdown
- $F_c$ Capillary Force
- $F^{SE}_c$ Capillary force reported by Surface Evolver
- $F_{cl}$ Contact line force density
- $F_{es}$ Force generated in electrostatic actuator
- $F_n$ $n$th term of the approximate equation of force
- $F_p$ Capillary force contribution from pressure
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_{SE}^p$</td>
<td>Capillary pressure contribution of force calculated by Surface Evolver</td>
</tr>
<tr>
<td>$F_{sat}$</td>
<td>Approximate maximum force achievable considering saturation</td>
</tr>
<tr>
<td>$F_{\sigma}$</td>
<td>Capillary force contribution from surface tension</td>
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<tr>
<td>$F_{SE}^{\sigma}$</td>
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<tr>
<td>$\tilde{F}_{\max}$</td>
<td>Approximate maximum force achievable</td>
</tr>
<tr>
<td>$f_{\max}$</td>
<td>Maximum force density</td>
</tr>
<tr>
<td>$\tilde{f}$</td>
<td>Nondimensional force</td>
</tr>
<tr>
<td>$f(R)$</td>
<td>Laplace equation for different liquid phase shapes</td>
</tr>
<tr>
<td>$g(R)$</td>
<td>Volume equation for different liquid phase shapes</td>
</tr>
<tr>
<td>$h$</td>
<td>Height of the bridge, or Plate separation in electrostatic actuators</td>
</tr>
<tr>
<td>$k$</td>
<td>Curvature of the surface on any arbitrary plane</td>
</tr>
<tr>
<td>$k$</td>
<td>Dummy variable used for definition of elliptical integrals</td>
</tr>
<tr>
<td>$k_{\max}$</td>
<td>Maximum curvature of a surface</td>
</tr>
<tr>
<td>$k_{\min}$</td>
<td>Minimum curvature of a surface</td>
</tr>
<tr>
<td>$\bar{k}$</td>
<td>Constant mean curvature of the bridge profile</td>
</tr>
<tr>
<td>$L_{SE}$</td>
<td>Perimeter of the wetted surface in Surface Evolver</td>
</tr>
<tr>
<td>$M$</td>
<td>Mean curvature operator</td>
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<td>$n$</td>
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</tr>
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<td>$n_{sat}$</td>
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<td>$\bar{n}$</td>
<td>Surface normal vector</td>
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<tr>
<td>$p$</td>
<td>Nondimensional capillary pressure</td>
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<td>$p^u, p^l$</td>
<td>Capillary pressure of upper and lower sections of the bridge in cavity configuration</td>
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<tr>
<td>$p_c^{SE}$</td>
<td>Pressure reported by Surface Evolver</td>
</tr>
</tbody>
</table>
\(P\)  
Uniform pressure applied to the membrane

\(P_a\)  
Partial pressure of air

\(P_b\)  
Approximate maximum pressure achievable considering breakdown

\(P_{\text{bias}}\)  
Bias term in pressure equation

\(P_c\)  
Pressure difference over the interface (capillary pressure)

\(P_l\)  
Pressure of liquid phase

\(P_{\text{sat}}\)  
Approximate maximum pressure achievable considering saturation

\(P_v\)  
Partial pressure of vapor

\(q\)  
Dummy variable used for definition of elliptical integrals

\(q_1, q_2\)  
Dummy variables used in analytical expressions of Delaunay sequence

\(R_d\)  
Electrical resistance of the dielectric

\(R_l\)  
Electrical resistance of the liquid

\(\mathcal{R}_v\)  
Ideal gas constant of vapor

\(r\)  
Bridge radius

\(r(z)\)  
Function representing the capillary bridge radius as a function of height

\(r^u, r^l\)  
Radius of upper and lower sections of the bridge in cavity configuration

\(r_0\)  
Radius of the neck or haunch in capillary bridges

\(r_1\)  
Other extremum radius along capillary bridge profile (not manifested physically)

\(r_c\)  
Radius of the contact line

\(r'_c\)  
Contact radius of the lower section of the cavity configuration

\(r_{\text{cyl}}\)  
Approximately constant radius of the bridge

\(r_{\text{mem}}\)  
Membrane radius

\(r_{\text{min}}\)  
Minimum radius of curvature
$r_{\text{max}}$  Maximum radius of curvature

$r_u^r, r_l^r$  Contact radii of a bridge with unequal contact angles

$r_u^u, r_l^u$  Contact radii of the upper section of the cavity configuration

$r_\perp$  Radius of curvature of the bridge surface’s profile

$s(\rho)$  Nondimensional lateral surface area of the bridge

$S_{\text{lg}}$  Liquid-gas surface area

$S_{\text{gl}}^u, S_{\text{gl}}^l$  Gas-liquid surface area of upper and lower sections of the bridge in cavity configuration

$S_{\text{ls}}$  Effective area of electrostatic actuator (parallel plate or comb configurations), or

Liquid-solid surface area on one side of the bridge

$S_{\text{ls}}^u, S_{\text{ls}}^l$  Liquid-solid surface area of upper and lower sections of the bridge in cavity configuration

$S_{\text{SE}}$  Area of the wetted surface in Surface Evolver

$S_s$  Total area of the solid (sum of solid-liquid and solid gas contributions)

$S_{xy}$  Area of the $xy$ two-phase interface

Subscripts $x$ and $y$ here take on the values ‘$s$’, ‘$l$’ and ‘$g$’ for solid, liquid, and gas respectively.

$T_0$  Temperature of the system

$t$  Dummy variable used for definition of elliptical integrals

$t_d$  Thickness of dielectric layer in CFA

$t_d^*$  Critical thickness of dielectric where saturation and breakdown happen simultaneously

$t_{\text{mem}}$  Membrane thickness

$U^\diamond$  Volume ratio for the pinned case

$V$  Total volume of the bridge
V\textsuperscript{u}, V\textsuperscript{l} Volumes of upper and lower sections of the bridge in cavity configuration

V\textsubscript{a} Volume of air

V\textsubscript{d} Volume of the dielectric between the electrode and the liquid

V\textsubscript{g} Volume of gas phase

V\textsubscript{v} Volume of vapor

v Voltage applied in electrostatic actuator

v(s) Laplace transform of voltage applied

v\textsubscript{b} Dielectric breakdown voltage

v\textsubscript{d} Voltage over dielectric layer

v\textsubscript{d}(s) Laplace transform of voltage across dielectric

v\textsubscript{sat} Saturation voltage

v\textsuperscript{*} Critical value of voltage applied

w(r) Deflection of a membrane as a function of radius

x Displacement in electrostatic actuator

x Dummy variable used for definition of elliptical integrals

Z\textsubscript{d} Parallel impedance for dielectric

z(x, y) Expression of fluid surface in Cartesian coordinates

\textbf{Greek}

\alpha Tilt angle

\chi Dummy variables used in analytical expressions of Delaunay sequence

\Delta F_{\text{max}} Maximum change in force achievable

\delta x Virtual displacement considered to find the force

E Total potential energy of the system

E\textsuperscript{u}, E\textsuperscript{l} Potential energy of upper and lower sections of the bridge in cavity
configuration

$E_{\text{cap}}$ Capacitive contribution to potential energy
$E_d$ Energy stored in dielectric layer
$E_{ex}$ Electrostatic potential energy
$E_{xy}$ Energy stored in $xy$ two-phase interface

Subscripts $x$ and $y$ here take on the values ‘$s$’, ‘$l$’ and ‘$g$’ for solid, liquid, and gas respectively.

$\varepsilon_0$ Permittivity of free space
$\varepsilon_d$ Permittivity of dielectric in CFA
$\varepsilon_r$ Relative permittivity of dielectric
$\varepsilon_{\text{mem}}$ Built-in residual strain of the membrane
$
\phi$
Dummy variable for integration with respect to contact angle, or Dummy variable used for definition of elliptical integrals
$
\phi_1, \phi_2$
Dummy variables used in analytical expressions of Delaunay sequence
$
\Gamma$
Measure of stiffness of membrane/ flexural rigidity
$
\varphi$
Angle between the profile tangent and the radial coordinate axis
$
\eta$
Dummy variable used for definition of elliptical integrals, or Dummy variable used in maximum deflection of membrane
$
\kappa$
Scale factor
$
\kappa^u, \kappa^l$
Scale factors of upper and lower sections of the bridge in cavity configuration

$\kappa^*$ Common value of scale factor for upper and lower sections of the bridge in cavity configuration
$
\mu$
Dummy variable used for definition of elliptical integrals, or Dummy variable used in maximum deflection of membrane
$
\nu$
Poisson’s ratio
$
\theta_0$
Contact angle of a liquid drop or bridge on a solid surface

$\theta_0(r_c)$ Contact angle as a function of contact radius
\(\theta^o_0(r), \theta^l_0(r)\) Contact angle gradient functions of upper and lower sections of the bridge in cavity configuration

\(\theta_i\) Inflection point contact angle

\(\theta_l, \theta_u\) Contact angles of a bridge with unequal contact angles

\(\theta_{in}\) Inner angle: Bridge angle facing the center of the profile at given point

\(\theta_{out}\) Outer angle: Bridge angle facing one of the bridge ends at given point

\(\theta_{sat}\) Saturation contact angle

\(\theta_v\) Apparent contact angle after electrowetting

\(\hat{\theta}(r_c)\) Interpolation function for contact angle in terms of contact radius for a symmetric bridge with fixed height and volume

\(P(\varphi, p)\) Function for nondimensional radius in terms of pair of nondimensional pressure and contact angle

\(\rho_d\) Electrical resistivity of dielectric

\(\rho_l\) Electrical resistivity of liquid

\(\rho_0\) Nondimensional radius of haunch or neck

\(\rho_i\) Nondimensional radius of second extremum along the profile

\(\rho_c\) Dimensionless contact radius

\(\rho_l, \rho_u\) Nondimensional contact radii or a bridge with unequal contact angles

\(\rho_w\) Density of water

\(\tilde{\rho}\) Dummy variable used for integration in terms of nondimensional radius

\(\sigma_{xy}\) Interfacial energy density (surface tension) at the \(xy\) two-phase interface

Subscripts \(x\) and \(y\) here take on the values ‘s’, ‘l’ and ‘g’ for solid, liquid, and gas respectively.

\(\tau_c\) Electrical time constant

\(\nu\) Dummy variable used for definition of elliptical integrals

\(\nu(\rho)\) Nondimensional bridge volume
\( \nu_j \) Function for nondimensional volume of a bridge with unequal contact angles for case \( j \)

\( \psi \) Angle between an arbitrary plane and the maximum curvature plane

\( \zeta \) Nondimensional bridge axial coordinate

\( \zeta_j \) Function for nondimensional height of a bridge with unequal contact angles for case \( j \)
1 Introduction

1.1 Capillary Force Actuators

Microactuators play a critical role in the development of microelectromechanical systems (MEMS), converting signals from electronic components into forces acting upon the physical world. They have found application in pumps and valves in micro total analysis systems (µTAS) and inkjet print heads, in optical switches and scanners, in variable capacitances and inductances for telecommunication, and in hard disk drive heads, among other devices. Although a variety of actuation technologies are available for MEMS, the force capability in each case results in relatively large actuators and limited stroke. Herein, an entirely new actuation approach for the microscale is proposed, capillary force actuation, which achieves significantly greater forces (10 – 100 times) than that of a similarly sized MEMS actuators when the voltage levels used are restricted to those commonly employed in integrated circuits.

Capillary force actuators (CFA) are an entirely new technology for microscale actuation [1]. CFAs operate on a novel principle. As illustrated in Figure 1.1, they employ a conducting liquid bridge between two surfaces. The surfaces contain electrodes covered with a very thin dielectric layer. Upon application of a voltage, the fluid’s apparent contact angle upon the surface changes, a phenomenon
previously observed and referred to as electrowetting on dielectric (EWOD). The change in contact angle results in a change in the bridge’s capillary pressure and therefore force acting on the surfaces. (Note that the dielectric layer prevents current flowing through the liquid.) While electrowetting has been used for moving fluids laterally along a surface, its potential for creating large forces normal to a surface has previously been unrecognized. The difference between CFA and EWOD droplet transport is not superficial; as we discuss below, the mechanisms of force production are fundamentally different.

In addition to significantly greater forces at low voltage, capillary force actuators possess other advantages over conventional choices for microscale actuation. The total movement achievable using a CFA is greater than that achieved with other actuators typically employed in MEMS. Because of its greater effectiveness, the voltage necessary for an actuation task is significantly lower (one tenth to one half) with CFA. Furthermore, out-of-plane forces can be easily achieved by CFA. Many optical and microfluidic applications require forces normal to the device plane, however, this is difficult to achieve with conventional MEMS actuators without sacrificing force capability or stroke.

1.2 Actuation for the Microscale – An Overview

A variety of actuation principles are available for MEMS. These may be classified in four families: electrostatic, electromagnetic, piezoelectric, and thermal [2]. Within each of these families, a variety of technologies are present, each technology having advantages and disadvantages with respect to force capability, power requirements, bandwidth, and device manufacture. Of particular importance to the applicability of an actuator
technology are the scaling properties associated with its physical principle. Those technologies that scale in proportion to length or area are generally better suited than those that scale in proportion to volume. It is generally recognized that despite the potential of very high energy densities, miniature electromagnetic actuators (including magnetostrictive) have relatively low force output in comparison to other approaches. Furthermore, fabrication challenges also impede their adoption. At present, MEMS piezoelectric actuators do not significantly exceed the force levels of electrostatic drives and generally do not provide as much stroke.

The most commonly used MEMS actuators belong to the electrostatic family, including comb, parallel plate (sometimes referred to as “lateral comb”), and scratch drive, among others. These actuators are generally simple and do not require special material or difficult-to-manufacture elements (e.g., ferrous cores or conductive coils). For small strokes (<1 \( \mu m \)), the parallel plate configuration, shown in Figure 1.2a, is usually employed. For larger strokes (1-100 \( \mu m \)), a comb configuration is preferred, see Figure 1.2b. Scratch drive actuators, in principle, can have unlimited displacements, but with much lower bandwidth (<1 Hz) than either comb drive or parallel plate (typically >10 KHz). The full actuation force capability of electrostatic actuators (dictated by breakdown voltage) is rarely realized in mainstream applications, as the voltages required (>500 V) would be much larger than is practical for integrated circuits. Therefore, the energy densities achieved in working devices are much lower than the theoretical numbers provided in the literature.
1.3 Electrostatic Actuators - Principle of Operation

To understand the benefits of capillary force actuators it is helpful to first consider the operational principles and limitations of electrostatic actuators. In both the parallel plate and comb configurations, the force generated follows the formulae

\[ F_{es} = \frac{1}{2} \left( \frac{\partial C}{\partial x} \right) v^2 \]

where \( C \) is the capacitance of the gap between the electrodes, \( S_{ls} \) represents actuators effective area, \( h \) is the gap length, \( v \) is the voltage applied, and \( \varepsilon_0 \) is permittivity of free space. The variation in the capacitance with displacement, \( \partial C / \partial x \), is due to different mechanisms in the two actuator configurations [3]. For the parallel plate configuration, the capacitance changes with displacement because the plate spacing, \( h \), is altered, see Figure 1.2a. As a consequence, this mechanism requires that the gap spacing must be larger than the stroke. (One advantage of the comb configuration is that it breaks this link between gap dimension and stroke, allowing the stroke to be many times the capacitive...
gap.) Unless the stroke needed is small (≈ 1 µm), the force achieved must be quite low as it is inverse quadratically dependent on gap

\[ \frac{\partial C}{\partial x}_{\text{PLATE}} = \frac{\varepsilon_0 S_{ls}}{h^2} \rightarrow F_{es} = \frac{1}{2} \left( \frac{\varepsilon_0 S_{ls}}{h^2} \right) v^2 \]  

(1.2)

It should be noted that the working stroke is often limited to one third of the plate spacing as pull-in instability prevents greater utilization.

With the necessary stroke specified, the designer has two avenues to increasing the achievable force of the parallel plate configuration, as Eqn. (1.2) indicates: larger voltage and greater plate area. The breakdown field strength of the gap ultimately limits the first of these. But, more importantly, the voltage level desirable or practical for an integrated circuit device (≈10-60 volts) is usually much lower than that prescribed by electrical breakdown. Since force achieved is proportional to voltage squared, this implies that the actual energy densities in the capacitive gap (equivalent to pressure) are far less than the theoretical values provided in the literature, which are based on breakdown field. Thus, the full force capabilities of these actuators are rarely realized.

For the comb configuration, the change in capacitance with displacement is due to a change in the capacitive area, \( S_{ht} \), see Figure 1.2b. As a result, the spacing between comb teeth, \( h \), may be made much smaller than the actuator stroke, boosting force capacity. In practice, manufacturing requirements typically limit the spacing to several microns. Achievable stroke for the comb configuration is limited by a lateral instability referred to as “side snap-over”. This instability results from the significant change in
capacitance that occurs with lateral motion. The avenues by which a designer might improve comb force capability (especially decreasing $h$) all result in a greater tendency toward lateral instability. (For reasons of space, we will not discuss the recently introduced vertical comb drive actuators [4]. CFA can be shown to possess superior force and stroke capabilities to these as well.)

This analysis provides some insight into the desirable properties for a MEMS electrostatic actuator:

- displacement resulting in a capacitance variation via a change in the capacitive area (similar to comb actuators), thus not directly limiting actuator stroke,
- negligible variation of capacitance with lateral displacement (similar to parallel plate actuators), eliminating lateral instability,
- a capacitive element with sub-micron thickness and high dielectric constant so as to greatly increase the capacitive energy stored and force produced.

In the next section, we will show that capillary force actuation possesses each of these properties, achieving them in a novel fashion.

1.4 Capillary Force Actuation: An Electrostatic Perspective

The basic principle of operation and the advantages of CFA may be understood by considering this technology in the same fashion as electrostatic devices. That is, an examination of the capacitive variation $\partial C/\partial x$ alone is particularly useful. Later, we will consider the electrohydrodynamic aspects of the actuation principle.
The capacitive element in CFA is a solid dielectric layer on the electrode surface. This layer can be manufactured so as to have a dielectric constant, \( \varepsilon_d \), that is 10 to 40 times greater than that of air, and a thickness, \( t_d \), a small fraction of the gap in comb drives. The area of the dielectric element that stores energy (and consequently determines capacitance) is that area wetted by the liquid bridge. The difference in electrical potential occurs chiefly across this dielectric layer since the liquid is chosen to be significantly more conductive than the dielectric layer. As the volume of liquid is conserved, a virtual displacement (\( \delta x \)) of one surface away from the other results in a change in wetted area

\[
\delta S_b = -\left(\frac{S_b}{h}\right)\delta x
\]

This implies a variation in the capacitance with displacement, specifically

\[
\left[ \frac{\partial C}{\partial x} \right]_{CFA} = \frac{\varepsilon_d S_b}{t_d h}
\]

where \( \varepsilon_d \) and \( t_d \) are the permittivity and thickness of the dielectric layer. A comparison of the CFA capacitance variation to that of parallel plate electrostatic actuator indicates the benefits of CFA. The ratio is

\[
\frac{\left[ \frac{\partial C}{\partial x} \right]_{CFA}}{\left[ \frac{\partial C}{\partial x} \right]_{PLATE}} = \left( \frac{\varepsilon_d}{\varepsilon_0} \right) \left( \frac{h}{t_d} \right)
\]

Since \( \varepsilon_d/\varepsilon_0 > 10 \) and \( h/t_d > 10 \), the force generated by CFA at a given (low) voltage will be more than one hundred times that of a parallel plate electrostatic actuator. (A similar conclusion holds for the comb configuration.)
1.5 Device Configuration

To initiate this investigation, the configuration considered will consist of two parallel electrodes, each covered by a thin hydrophobic, dielectric layer, as shown in Figure 1.1. Between the two plates lies a capillary bridge of conductive fluid (conductive in comparison to the dielectric layer). One or both electrodes is held by flexible supports which allow motion toward or away from the other but constrain the motion otherwise. It is assumed for this analysis that the electrodes will remain parallel to each other. A detailed model of this device is developed in Chapter 2. In Chapter 5, alternative configurations of the actuator are examined.

1.6 Fundamental Results on Capillary Surfaces

Surface tension is the work $\delta w$ required to increase the surface area by an infinitesimal quantity $\delta S$. This concept was introduced by Segner [5] in 1751. Later, Young [6] and Laplace [7] independently reported a proportional relation between the curvature of a meniscus and the resulting force on a fluid volume. In 1830 Gauss [8] derived the capillary equation using an energy minimization approach, considering both gravity and rotation. The axisymmetric solutions of the capillary equation in the absence of gravity and rotation were developed by Delaunay [9] in 1841 and are now named after him, the Delaunay curves. In 1873 Plateau [10] in classifying the set of meniscus shapes, introduced a new parameter, the dimensionless capillary pressure $p$ which will be particularly important to this investigation.
When three phases are present, as in the case of capillary bridges, the surface energies between the phases are critical to the bridge shape. We shall denote the interfacial energy density (surface tension) at the $xy$ two-phase interface as $\sigma_{xy}$ and the area of the interface as $S_{xy}$. The subscripts $x$ and $y$ here take on the values ‘$s$’, ‘$l$’ and ‘$g$’ for solid, liquid, and gas respectively. These interfacial energies determine the contact angle of the liquid on the solid, $\theta_0$, as described by the Young-Dupré equation:

$$\sigma_{gl} \cos(\theta_0) = \sigma_{gs} - \sigma_{ls}$$ (1.6)

or

$$\cos(\theta_0) = \frac{\sigma_{gs} - \sigma_{ls}}{\sigma_{gl}}$$ (1.7)

The Young equation may be formulated as a balance of interfacial forces or as a minimum in total interfacial energy.

Furthermore, surface tension gives rise to a pressure difference $P_c$ between the gas and liquid phases. Any continuous and 2nd order differentiable surface may be described in the vicinity of a point by the tangent plane and the two principle curvatures. Principle curvatures are the maximum ($k_{max} = 1/r_{max}$) and the minimum
(\( k_{\text{min}} = 1/r_{\text{min}} \)) curvature observed on two planes perpendicular to each other and to the tangential plane. (Figure 1.4)

Curvature of the surface on any plane perpendicular to the tangential plane may be calculated as:

\[
k = k_{\text{max}} \cos^2(\psi) + k_{\text{min}} \sin^2(\psi)
\]

where \( \psi \) is the angle this plane makes with the maximum curvature plane.

Considering the pressure difference and the surface tension, the balance of forces for an element of fluid yields:

\[
2\sigma_{\text{gl}} \sin(\frac{d\psi_1}{2}) r_2 d\psi_2 + 2\sigma_{\text{gl}} \sin(\frac{d\psi_2}{2}) r_1 d\psi_1 = P_c r_1 d\psi_1 r_2 d\psi_2
\]

Simplification gives us the Young-Laplace Equation:

\[
P_c = \sigma_{\text{gl}} \left( \frac{1}{r_2} + \frac{1}{r_1} \right)
\]

where \( P_c \) is the difference in pressure over the interface and includes contributions from static pressure (caused by gravity and/or rotation) and dynamic pressure (such as caused by inertial forces during oscillation).

As with any other physical phenomenon where no irreversible energy loss is involved the results obtained by force balance may alternatively be derived by the minimization of the total energy. One may express the fluid surface in Cartesian coordinates as \( z(x, y) \).

The area, \( A \), of the fluid surface and the liquid volume, \( V \), may be obtained as:

\[
S_{\text{gl}} = \iint \sqrt{1 + \left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2} \, dx \, dy
\]

(1.11)
\[ V = \iint z(x, y) \, dx \, dy \]  
(1.12)

Considering the case of no rotation or gravity, the total energy of the fluid is given by:

\[ E = \sigma_{gl} S_{gl} \]  
(1.13)

Minimization of energy under fixed volume constraint may also be interpreted as minimization of \( E - P_c (V - V_0) \) where the capillary pressure is the Lagrange multiplier.

According to the calculus of variations, minimization of \( E \) is achieved for

\[ P_c = \sigma_{gl} \left( \frac{dn_x}{dx} + \frac{dn_y}{dy} \right) \]  
(1.14)

where \( n_x \) and \( n_y \) are the components of the surface normal \( \vec{n} \).

\[ \vec{n} = (n_x, n_y, n_z) = \frac{1}{\sqrt{1 + (\partial z / \partial x)^2 + (\partial z / \partial y)^2}} \left( -\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1 \right) \]  
(1.15)

This is equivalent to Eqn. (1.10).

### 1.7 Electrowetting

In 1875 Lippmann [11] introduced the concept of Electrocapillarity, the basis for modern electrowetting. He succeeded in varying the capillary depression of mercury in contact with electrolyte solutions by applying a voltage between the mercury and the electrolyte.

Later, Berge [12] initiated research on electrowetting on dielectric (EWOD) by introducing the use of an insulating layer between liquid and electrode to prevent electrolysis of the liquid while applying voltage. In a generic EWOD setup, a partially wetting liquid is used on top of a planar solid electrode covered with a thin layer of dielectric. Applying voltage between the liquid and the electrode is shown to increase the
wetting of the liquid resulting in lower apparent (macroscopic) contact angles (Figure 1.5).

Electrowetting theory may be explained via three different approaches: the thermodynamic and electrochemical approach, the energy minimization approach, and the electromechanical approach [13]. In Lippmann’s classical electrochemical approach he considers metal-electrolyte interfaces and examines the effect of a double ionic layer on the effective interfacial tension. This approach may be generalized to yield the Young-Lippmann equation. In the energy minimization method one may consider the total potential energy of the system as the sum of interfacial energy contribution and electrostatic contribution. Comparison of this energy with the free energy in absence of electric potential yields the Young Lippmann equation [14]. This is the approach we will employ later in Chapter 2 to find the apparent contact angle of the capillary bridge in our device. Finally, the electromechanical approach examines the Maxwell stress on the liquid-gas interface in the presence of an electric field. Integration of this stress yields the force acting upon the surface, which may be considered as localized at the contact line, and also produces the Young-Lippmann equation. More detailed explanation of these three approaches may be found in a recent review paper on electrowetting research [13].
1.8 Contact Angle Saturation

Contact angle saturation is a phenomenon where the liquid will not fully wet a dielectric coated electrode. The contact angle achievable under EWOD is limited even if very high voltage is used. At present, six different mechanisms have been proposed as the cause of contact angle saturation, most of which are charge leakage mechanisms due to the strong electric field near the contact line. Here we will discuss these mechanisms.

**Dielectric charge trapping** is a mechanism that was proposed by Verheijen and Prins [15] in 1999. They pointed out the possibility of charges being trapped in or on the dielectric layer when there is a stronger interaction between ions and the dielectric than between ions and the liquid. They derived a modified version of the Young-Lippmann equation in which a term for the potential of a trapped charge layer outside the liquid is considered. The authors calculated the potential function numerically based on their experimental results and showed the independence of this function from polarity of the applied voltage, the ion type, ion molarity, and ion valence. However, it was not possible to point to an underlying charge bonding mechanism for the proposed behavior of charges. Furthermore, the mechanism posits these trapped charges outside the wetted surface area, a situation which seems difficult to explain.

**Air ionization** in the vicinity of contact line is another mechanism proposed as a potential cause of the saturation effect. Vallet et al. [16] reported a series of optical emission pulses and current spikes starting at the same voltage that saturation is observed. It has been further argued that air ionization close to the contact line makes the effect of electrical
stress on the surface less efficient, and hence less than the expected amount of wetting would be observed. In addition, air ionization has been considered as a potential cause for the irreversible hydrophilicity on a narrow ring around the drop that is sometimes observed. In this scenario, high electric field near the drop edge would be responsible for the air ionization effect.

Vallet et al. [16] also proposed another mechanism for low conductivity liquids, the instability of the contact line and the ejection of satellite drops. Due to the high density of same-polarity charges at the contact line with high voltage and the strong repulsion between them, the liquid surface may become unstable, expelling small droplets. Vallet et al. performed a linear stability analysis by assuming the contact line to have a sinusoidal perturbation. The electrostatic energy gain due to increase in the length of contact line as well as change in charge distribution close to contact line was determined and compared to the surface energy cost due to change of drop shape close to contact line. In this manner, Vallet et al. found expressions predicting the stability limit as observed in their experiments, suggesting that this model contains the essential physics. However, this model failed to explain the suppression of instability that occurred upon addition of salt to the liquid.

Peykov et al. [17] and Quinn et al. [18] proposed that the Young-Lippmann equation fails to predict the apparent contact angle when zero effective liquid-solid surface tension was reached. They conjectured that for potentials greater than that which would produce the critical value (i.e., negative values of effective liquid-solid surface tension) the interface
becomes thermodynamically unstable. Although some experimental evidence supports this conjecture, the limit proposed by Ralston et al. does not have strong theoretical support and has not received broad acceptance as a result.

Shapiro et al. [14] showed that electrical resistivity of the liquid could cause contact angle saturation in electrowetting of sessile drops. The drop shape was assumed to stay as a section of sphere and the partial differential equation for electrostatic field was solved numerically to find the resistance of the liquid as a function of contact angle. Energy minimization was then used to predict the contact angle achieved under electric field for a range of applied voltage. The results seem to match with the published experimental results from several groups. However, the mechanism proposed holds only for low resistivity ratios $\rho_d t / \rho_l R$ ($\rho_d$ and $\rho_l$ denote the resistivity of dielectric and liquid respectively, $t$ and $R$ denote the dielectric thickness and drop radius respectively) and does not explain results achieved with highly conductive liquids.

Finally, Papathanasiou and Boudouvis [19] in 2005 emphasized the correlation between saturation and local electrical breakdown of the dielectric. A sessile drop on an electrode covered with a dielectric was considered and the Young-Laplace equation and the Laplace equation for the electrostatic potential were solved simultaneously under the constant-drop-volume condition. The shape of the drop and electric field distribution were computed numerically. The field strength was compared to the dielectric breakdown strength. It was suggested that electrowetting is limited by the voltage where the maximum field strength reaches the dielectric breakdown strength. A strong agreement
between the numerical results and the available experimental data was shown. Later, the same team conducted a set of experiments in which different dielectric materials were used and the leakage current was recorded while the voltage was increased [20]. It was shown that the saturation is accompanied by a significant increase in the leakage current through the dielectric. Also, at the onset of the saturation, a linear dependence between the leakage current and drop radius was reported, suggesting that leakage was occurring in a ring-like strip adjacent to contact line. For voltages higher than the saturation voltage the leakage current was found to approximately scale with the square of drop radius suggesting breakdown of dielectric throughout the entire wetted area. It was argued that breakdown restrains the polarization of the dielectric and thus limits the electric field strength. This reduces the effect of electrical stress close to the contact line and therefore results in less than predicted apparent contact angles.

The relatively high number of mechanisms proposed to explain contact angle saturation and the number of conflicting experimental results suggests that there is no one mechanism responsible for all the experimental observations. Nevertheless, high intensity of the electric field at the contact line is thought to be the origin of the more extensively accepted mechanisms (i.e. charge trapping, instability of the contact line, and local dielectric breakdown). When exploring actuator design limits and optimization in this dissertation, the equations for saturation will be expressed in a fashion so as to be independent of the origin of the saturation phenomenon. Rather, the saturation limit will be written in terms of the change in cosine of contact angle before saturation is observed.
It is our belief that at this time saturation limit of a particular device may only be determined by conducting electrowetting experiments.

1.9 Comparison to Electrowetting Droplet Transport

The capillary force is the Laplace pressure multiplied by the wetted area. This quantity is different from the electrode-normal component of liquid/vapor surface tension along the contact line. This latter term also contributes to actuation in CFA, but is typically 5 – 10 times smaller than the capillary pressure term. This distinction is particularly important when considering the differences between CFA and electrowetting-based droplet transport. In droplet transport [21, 22], the actuation force arises from an imbalance of the contact line integral of effective surface tension on opposite sides of the droplet; the Laplace pressure does not contribute to the transport force [23, 24]. To illustrate the difference, consider a 250 \( \mu m \) radius sessile water drop undergoing electrowetting-on-dielectric transport. The maximum transport force exerted on the drop will be about 40 \( \mu N \). A CFA using the same voltage, with a 250 \( \mu m \) radius 50 \( \mu m \) height capillary bridge, exerts a capillary force of over 340 \( \mu N \).
2 Actuator Modeling

2.1 Standard Configuration

The standard configuration of CFA consists of a conducting liquid bridge between two parallel surfaces. The surfaces contain electrodes covered with a very thin dielectric layer (Figure 2.1). Symmetric boundary conditions result in an axisymmetric bridge profile. The volume of drop is assumed to be constant and is denoted by $V$. We will show that for given bridge volume $V$, distance between the two surfaces, $h$, and the contact angle of the liquid on the surfaces, $\theta_0$, the shape of bridge and the force exerted can be uniquely determined.

2.2 Device Physics

For the scale considered, gravity may be ignored as it has negligible effect. For example, the capillary force exerted by a 250 $\mu m$ radius, 50 $\mu m$ height capillary bridge of water is over 340 $\mu N$ while its weight is less than 0.1 $\mu N$. The bridge is confined in a closed environment and evaporation losses are negligible (see Appendix A for brief analysis of evaporation). In static equilibrium without applied electric field, the energy is that of the surfaces alone and the capillary bridge satisfies Laplace’s equation (constant mean
curvature) subject to the boundary condition at the contact line dictated by Young’s equation,
\[ \sigma_{gs} - \sigma_{ls} - \sigma_{ls} \cos(\theta_0) = 0 \] (2.1)
which expresses the contact angle of the liquid \( \theta_0 \) in terms of the interfacial energies between the liquid, solid, and vapor phases [25].

In most cases for static analysis, we may consider the liquid bridge as perfectly conducting. From a device performance perspective, high liquid conductivity aids both actuator gain and bandwidth, as a simple equivalent circuit model indicates (see Appendix B). With the wide range of liquids and dielectrics available, liquid conductivity 5 to 15 orders of magnitude greater than that of the dielectric layer may be achieved. Therefore, such an assumption is usually accurate in predicting quasi-static behavior.

When voltage is applied, the electric field results in electrohydrodynamic forces on the fluid, as outlined herein. The electric potential results in a build up of a charge double layer at the dielectric / electrode interface and the dielectric / liquid interface, with free charge in the conducting bridge and electrode, and polarization charge in the dielectric [24, 26]. The capacitance of this arrangement will be essentially equal to that of the dielectric layer itself. From an electromechanical viewpoint, the presence of free charge in the liquid, interacting with the electric field, imparts electrohydrodynamic forces to the fluid [26]. In the perfectly conducting case, free surface charge in the liquid bridge screens the electric field \( \vec{E} \) from the bridge’s interior and charge only appears at the liquid surface. In this case, the tangential component of the electric field is zero at the
liquids surface [13]. Denote the normal component as $E_n$. The force acting upon an infinitesimal portion of the surface $dS_{gl}$ is directed along the outward normal with a pressure given by:

$$P_E = \varepsilon_0 E_n^2 / 2$$  \hspace{1cm} (2.2)

The normal electric field, $E_n$, at the liquid-vapor interface is negligible except very close to the contact line. Excluding that region, we may state that no surface force acts upon the liquid-vapor interface in the case of a conducting bridge. Along the solid-liquid interface $E_n$ is uniform, with the exception of near the contact line. As the contact line is approached on either interface, the charge density and field increase rapidly [16]. On the scale of the device considered, the three-phase contact line may be treated as one-dimensional. In this spirit, integration of Maxwell’s stress tensor in the vicinity of the contact line yields a contact line force (per unit length). It has been shown [16] that the component tangential to the solid surface is the force density (per unit length) given by

$$F_{cl} = \frac{\varepsilon_d}{2t_d} v_d^2$$  \hspace{1cm} (2.3)

where $v_d$ is the electrical potential across the dielectric layer. A balance of this contact line force, and the interfacial tensions acting upon the contact line yields the Lippmann-Young equation,

$$\cos(\theta_c) = \cos(\theta_0) + \frac{\varepsilon_d}{2\sigma_p t_d} v_d^2$$  \hspace{1cm} (2.4)

which describes the change in contact angle with voltage [24, 13]. From a modeling perspective, we may either employ the Lippmann-Young relation at the equilibrium state or place the contact line force density in the momentum equation, as these are equivalent.
The decrease in contact angle due to the applied voltage results in a change in capillary bridge shape. As this is the only effect of the electric field, the new bridge shape profile will belong to the Plateau sequence, the family of curves that are the known axisymmetric solutions to the Laplace equation (unduloids, nodoids, catenoids, and cylinder) [27]. A detailed analysis is presented in Section 2.2.1. In the case of ‘pancake’ bridges, a simple analysis is available and illustrative. Laplace’s equation states that the capillary pressure is related to the bridge radius, \( r_{cyl} \), and the radius of curvature of the bridge surface’s profile, \( r_{\perp} \), via

\[
P_c = \sigma_{gl} \left[ \frac{1}{r_{cyl}} + \frac{1}{r_{\perp}} \right]
\]  

(2.5)

For pancake bridges, it is reasonably accurate to approximate the profile as a circular arc. In this case,

\[
P_c = \sigma_{gl} \left[ \frac{1}{r_{cyl}} - 2 \cos(\theta_0)/h \right]
\]  

(2.6)

Employing Eqn. (2.4) yields the relationship between voltage across the dielectric and the capillary pressure acting on the solid surface:

\[
P_c = P_{bias} - \frac{\varepsilon_d v_d^2}{t_d h}
\]  

(2.7)

\[
P_{bias} = \frac{\sigma_{gl}}{r_{cyl}} \left[ \frac{2 \sigma_{gl} \cos(\theta_0)}{h} \right]
\]  

(2.8)

Thus, application of voltage results in a change in capillary pressure, hence force acting between the electrodes. Note that the ‘bias’ term \( P_{bias} \) will be small in comparison to the variation in pressure. In fact, \( P_{bias} \) will be equal to zero for a value of contact angle \( \theta_0 \) near 90°.
2.3 Capillary Bridge Shape and Force without Electric Field

To begin our analysis of the device, we will consider the determination of the force exerted between two surfaces by a capillary bridge between them. For this analysis, we will assume that the device size is such that gravity may be ignored and that the surfaces and alignment of them are ideal. In this case, the bridge shape will be axisymmetric and one of the Plateau sequence of shapes. Assuming that the volume of the bridge, $V$, the distance between the two surfaces, $h$, and the contact angle of the liquid on the surfaces, $\theta_o$, are given, the shape of bridge and the force exerted can be uniquely determined. With the bridge shape known, the force exerted by it may be calculated by a variety of semi-analytic and numerical methods.

2.3.1 Bridge Shape

Define the aspect ratio of the device as

$$a = \sqrt{\frac{\pi h^3}{4V}}$$

(2.9)

This dimensionless parameter will enable a detailed scaling analysis, to be performed later. It can be also easily shown that in the case of a cylindrical bridge $a = h/D$, where $D$ is the diameter of the cylinder. Thus, a cylindrical bridge of equal diameter and height will have $a = 1$. With aspect ratio fixed, $h$ may be employed as a scaling factor.

Delaunay [9] found that the shapes of capillary bridges that extend between two parallel plates may be parameterized by the two radii at which the Delaunay curve achieves a tangent line parallel to the central axis of the bridge. One of these radii, $r_o$, will be the
radius of the neck or haunch that lies between the two plates. (If the plates have the same contact angle as we are now considering, \( r_0 \) will be the radius at the mid point between the plates.) The other radius, \( r_1 \), is not manifested physically for a stable bridge. Rather, it occurs if the Delaunay curve is continued beyond the physical confines of the liquid. This is illustrated in Figure 2.2.

![Figure 2.2: Important radii and contact angle in parameterization of an unduloid profile](image)

In the case of equal contact angles \( \theta_0 \) on both surfaces, a particular bridge is specified by providing two of the following values: bridge volume (\( V \)), radius of the contact line (\( r_c \)), and spacing between the plates (\( h \)). Note that the contact angle is the angle of Delaunay curve tangent at the three-phase line. The solution to the boundary condition problem would be unique if we only consider the stable capillary bridges.

The non-dimensional capillary pressure, which Plateau introduced to categorize the bridge shape, is defined as
where the capillary pressure $P_c$ is the difference between bridge interior and exterior pressures. The Plateau sequence of shapes consists of nodoids and unduloids with the special cases of sphere, cylinder and catenoid. The type of the profile may generally be parameterized by the value of Kralchevsky’s dimensionless pressure $p$. Table 2.1 summarizes this classification.

Table 2.1: Classification of Plateau sequence of shapes parameterized by $p$

<table>
<thead>
<tr>
<th>Bridges with Neck</th>
<th>Bridges with Haunch</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p &lt; 0$ Nodoid</td>
<td>$\frac{1}{2} &lt; p &lt; 1$ } Unduloid</td>
</tr>
<tr>
<td>$p = 0$ Catenoid</td>
<td>$p = \frac{1}{2}$ Cylinder</td>
</tr>
<tr>
<td>$0 &lt; p &lt; \frac{1}{2}$ Unduloid</td>
<td>$\frac{1}{2} &lt; p &lt; 1$ Unduloid</td>
</tr>
<tr>
<td>$p = 1$ Sphere</td>
<td>$1 &lt; p$ Nodoid</td>
</tr>
</tbody>
</table>

Since these are the only solutions to the axisymmetric Laplace equation, the profile of any capillary bridge must be realized as a scaled version of a section of one of these shapes. The analytical expressions for Plateau sequence are listed in Table 2.2. Detailed derivation of analytical expressions of bridge shape is included in Appendix C. An interesting fact about the unduloids is the symmetry among their profiles around the cylinder case [see Appendix D for more details].
In Table 2.2, $\rho$ is the nondimensional profile radius; $\zeta$ is the nondimensional bridge axial coordinate with $\zeta = 0$ corresponding to nondimensional radius $\rho_0$; $\varphi$ is the angle between the profile tangent and the radial coordinate axis; and $\nu(\rho)$ is the nondimensionalized bridge volume between $\zeta = 0$ and $\zeta = \zeta(\rho)$. The physical values of the radii, axial coordinate, and volume may be calculated via:

$$r_0 = \kappa \rho_0, \quad r_1 = \kappa \rho_1, \quad r_c = \kappa \rho_c, \quad r = \kappa \rho, \quad z = \kappa \zeta, \quad V = \kappa^3 v$$  \hspace{1cm} (2.11)$$

The nondimensional radii $\rho_0$ and $\rho_1$ are related to the dimensionless pressure as:

$$\rho_0 = |p|, \quad \rho_1 = |1 - p|$$  \hspace{1cm} (2.12)$$

Table 2.1 was previously provided in part in Kralchevsky and Nagayama [27]. However, here all the expressions are presented in nondimensional format. Note that all the functions introduced in this table have implicit dependence on nondimensional pressure.

<table>
<thead>
<tr>
<th>Bridges with Neck</th>
<th>Bridges with Haunch</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_0 \leq \rho \leq \rho_1$</td>
<td>$\rho_1 \leq \rho \leq \rho_0$</td>
</tr>
<tr>
<td>$p &lt; 0$</td>
<td>$p = 0$</td>
</tr>
<tr>
<td>$0 &lt; p &lt; \frac{1}{2}$</td>
<td>$p = \frac{1}{2}$</td>
</tr>
<tr>
<td>$\frac{1}{2} &lt; p &lt; 1$</td>
<td>$p = 1$</td>
</tr>
<tr>
<td>$1 &lt; p$</td>
<td>$\chi = -1$</td>
</tr>
<tr>
<td>Nodoid</td>
<td>Catenoid</td>
</tr>
<tr>
<td>$\chi = +1$</td>
<td>Unduloid</td>
</tr>
<tr>
<td>Cylinder</td>
<td>Unduloid</td>
</tr>
<tr>
<td>Sphere</td>
<td>Nodoid</td>
</tr>
<tr>
<td>$\chi = -1$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Notation</th>
<th>$q_1 = \sqrt{1 - \rho_0^2 / \rho_1^2}$</th>
<th>$q_2 = \sqrt{1 - \rho_1^2 / \rho_0^2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sin(\phi_1) = \frac{\sqrt{1 - \rho_0^2 / \rho_1^2}}{q_1}$</td>
<td>$\sin(\phi_2) = \frac{\sqrt{1 - \rho_1^2 / \rho_0^2}}{q_2}$</td>
<td></td>
</tr>
</tbody>
</table>
The equations in Table 2.2 may also be used for nonsymmetric bridges with some modifications to the equations of volume and area as mentioned in Chapter 5. Given contact angle and nondimensional pressure, dimensionless contact radius, $\rho_c$ may be calculated as:

$$\rho_c = \text{P}(\theta_0, p)$$  \hspace{1cm} (2.13)

where $\text{P}(\varphi, p)$ gives the nondimensional radius for given pair of nondimensional pressure and contact angle and is defined as:

$$\text{P}(\varphi, p) = \sqrt{\rho_0 \rho_1 + \frac{1}{2} \left[ \sin^2(\varphi) \pm \sqrt{(\sin^2(\varphi) + 4 \rho_0 \rho_1 \sin^2(\varphi))} \right]}$$  \hspace{1cm} (2.14)
where plus or negative sign is used based on whether a bridge with haunch or neck is
analyzed respectively. Scale factor $\kappa$ may be found by equating the profile height $z(\rho)$
at contact line with one half of actual height of the bridge as:

$$z(\rho_c) = \kappa \zeta(\rho_c, p) = \frac{h}{2}$$

(2.15)

Solving for $\kappa$ yields:

$$\kappa = h / [2 \zeta(P(\theta_0, p), p)]$$

(2.16)

The volume of the bridge may be calculated as:

$$V = \kappa^3 \nu(\rho_c, p) = \kappa^3 \nu(P(\theta_0, p), p)$$

(2.17)

Substitution for $\kappa$ from the equation above gives the equation for the bridge volume in
terms of dimensionless pressure, contact angle and bridge height:

$$V = \frac{h^3 \nu(P(\theta_0, p), p)}{[2 \zeta(P(\theta_0, p), p)]^3}$$

(2.18)

Using the equation above for given bridge height, volume and contact angle one may find
the appropriate dimensionless pressure by employing a one dimensional root finding
process. This equation may also be written in terms of aspect ratio $a$:

$$a = \sqrt[3]{\frac{\pi h^3}{4V}} = \sqrt[3]{\frac{2\pi \zeta(P(\theta_0, p), p)}{\nu(P(\theta_0, p), p)}}$$

(2.19)

which indicates that the value of the dimensionless pressure obtained by solving this
equation is only dependent on aspect ratio and is independent of scaling.

Having the value of dimensionless pressure, the parameters governing the bridge shape
$r_0, r_i, r_c, r$, and $z$ may be calculated as scale factor $\kappa$ multiplied by the values of the
nondimensional parameters $\rho_0, \rho_i, \rho_c, \rho$, and $\zeta$ respectively. Figure 2.3 illustrates the
change in the bridge shape with the change of contact angle for a drop with fixed volume and height. It may be observed that the higher contact angle will result in higher dimensionless pressure and smaller radius for the wetted area.

2.3.2 Force Exerted

Given the bridge shape parameters for a given device configuration, the axial force may be determined in semi-analytic fashion via two approaches: (1) differentiation of the total potential energy with respect to a virtual displacement; and (2) summation of the capillary pressure and surface tension contributions. (Due to axial symmetry, the transverse force produced will be zero.) Force may also be determined via an entirely numerical approach; this will be examined in Section 3.6.

**Energy Approach.** The force exerted by the bridge may be determined by differentiating the potential energy stored in the device with respect to a virtual displacement of one surface. With no electric field, the energy stored is the sum of three surface energies, each the product of the surface tension coefficient and the area of the two phase surfaces:

\[
E = 2\sigma_{gs} S_{gs} + \sigma_{gl} S_{gl} + 2\sigma_{ls} S_{ls}
\]  

(2.20)
The sum of the solid-gas and liquid-solid areas is equal to the total area of solid and is therefore constant:

\[ S_{gs} + S_{ls} = S_s \]  

(2.21)

Combining Eqns. (2.20) and (2.21), the total energy may be written as:

\[ E = \sigma_{gI}(S_{gl} - 2S_{ls} \cos(\theta_0)) + 2\sigma_{sl}S_s \]  

(2.22)

Since the last term is constant, it will not contribute to the force exerted. Define

\[ E_{gl} = \sigma_{gl}S_{gl} \]  

(2.23)

\[ E_{ls} = (-2\sigma_{gl} \cos(\theta_0))S_{ls} \]  

(2.24)

The capillary force, \( F_c \), may be written as:

\[ F_c = -\frac{d}{dh}(E_{gl} + E_{ls}) = -\sigma_{gl} \frac{dS_{gl}}{dh} + 2\sigma_{gl} \cos(\theta_0) \frac{dS_{ls}}{dh} \]  

(2.25)

To evaluate, the variation of the areas \( S_{gl} \) and \( S_{ls} \) with bridge height, \( h \) must be determined. The areas \( S_{gl} \) and \( S_{ls} \) may be found via

\[ S_{ts} = \pi r_c^2 \]  

(2.26)

\[ S_{gl} = 2 \int_0^{h/2} 2\pi r(\zeta) \sqrt{1 + \left(\frac{dr}{dz}(\zeta)\right)^2} d\zeta \]  

(2.27)

Here, we have avoided using the tangent slope \( \frac{dz}{dr}(r) \) in the expression for \( S_{gl} \) as it would result in significant numerical difficulties for values of \( z \) near 0. Calculation of \( S_{gl} \) using the equation above would be only feasible if radius can be expressed as a function of bridge height \( r(z) \), that is the inverse of the function \( z(r) \) introduced.
previously. Because of the complexity of an analytical inversion of \( z(r) \), the inverse function is constructed numerically as an interpolation of discrete sampled points.

Using these expressions, the areas may be calculated for given values of bridge height \( h \) and contact angle \( \theta_0 \). For each value of \( \theta_0 \), a polynomial curve in terms of independent variable \( h \) may be fitted to these numerical results and the terms needed may be found by analytically differentiating \( S_{gl} \) and \( S_{ls} \) with respect to \( h \).

**Stevin Approach.** A complimentary method for calculation of the force follows the classical approach introduced by Stevin [27]. The force exerted by the bridge is expressed as the sum of contributions from the surface tension \( \sigma_{gl} \) and the capillary pressure, \( P_c \):

\[
F_\sigma = -2\pi r_0 \sigma_{gl}
\]

(2.28)

\[
F_p = \pi r_0^2 P_c
\]

(2.29)

\[
F_c = F_\sigma + F_p = -2\pi r_0 \sigma_{gl} (1 - p)
\]

(2.30)

Given contact angle \( \theta_0 \) and bridge volume \( V \), Eqn. (2.18) may be solved for nondimensional pressure \( p \). The radius \( r_0 \) may then be found using Eqn. (2.11). Finally, the force may be determined using Eqn. (2.30). Approximate expressions may be derived for low aspect ratio bridges (see Appendix E) for which the equivalence of Eqns. (2.25) and (2.30) may be shown (see Appendix F).

The nondimensional pressure, \( p \), is an indicator of the sign of the capillary force. An attractive force between the surfaces occurs when \( p < 1 \). When \( p > 1 \) a repulsive force is
exerted by the bridge. In the special case of \( p = 1 \) the bridge is a portion of a sphere and zero force results.

**Discussion:** It should be noted that capillary forces exerted by liquid bridges can be very large, many times greater than those generated by electrostatic devices. For example, we can compare the capillary pressure exerted between two hydrophilic surfaces \( (\theta_0 = 45^\circ) \) placed 2 \( \mu \text{m} \) apart to the electrostatic pressure exerted by a MEMS parallel plate actuator with the same spacing. To match the capillary pressure of a water bridge, the electrostatic actuator would require 215 volts. For 10 \( \mu \text{m} \) spacing, the voltage needed by an electrostatic actuator would be 480 volts. If the surfaces were separated by 50 \( \mu \text{m} \), over 1000 volts would be needed to match the capillary force.

### 2.4 Capillary Bridge Shape and Force with Electric Field

#### 2.4.1 Bridge Shape

To this point, we have examined a capillary bridge without any applied electric field. We now turn our attention to the case where an electric potential is applied between the two electrodes. As we shall see, application of voltage to the device results in a change in contact angle and hence capillary bridge shape. Here we first discuss the effects applied electric field have on the bridge shape and force and then use an energy minimization approach to find the apparent contact angle under applied voltage.
For over a decade, electrowetting phenomenon was explained in terms of a reduction in the interfacial energy of the liquid-solid interface with applied voltage. The balance of the surface tensions at the contact line was thereby altered, resulting in a lower contact angle. Lately, another view of electrowetting phenomenon has been introduced in which electrowetting is viewed as a purely electromechanical effect. As discussed in Section 2.2, it has been proposed that the applied electric field causes an additional electrostatic pressure on the liquid surface and the change in apparent contact angle is a result of pressure balance on the liquid surface rather than a force balance at the contact line, see Figure 2.4. It has further been shown by Mugele et al. [28, 29, 23] that under an electric field, the local contact angle remains unchanged and only the apparent contact angle is altered as captured by the Young-Lippmann equation:

$$\cos(\theta_v) = \cos(\theta_o) + \frac{\varepsilon_d}{2\sigma_{gl} t_d} \nu_d^2$$  \hspace{1cm} (2.31)

Figure 2.4: Close-up of bridge profile near the contact line: (a) former model, (b) recent model.
The Maxwell stress tensor was first used by Jones [30] to calculate the net electrostatic force on the profile surface. In this manner the effect of electric field was regarded as an electrostatic pressure on the liquid profile given by:

$$P_E = \varepsilon_0 E_n^2 / 2$$  \hspace{1cm} (2.32)

It was further shown by Vallet et al. [16] that this pressure is only significant near the contact line, within a distance of a few times the dielectric thickness. In fact, this is the region that the transition from the native contact angle to the apparent contact angle occurs.

Jones [31] was also first to point out that the change in apparent contact angle and the force exerted on the liquid are two independent phenomena (although both are produced by the Maxwell stress). Today, it has been established that the change in contact angle can be explained by the distribution of the electrostatic force close to the contact line and is independent of the liquid shape (sessile drop, capillary bridge). On the other hand the change in the liquid shape and possible transport of the liquid may be explained in terms of balance of the pressure inside the liquid and the net electrostatic force exerted on the liquid profile.

Figure 2.5: Capillary bridge profile with (solid lines) and without (dashed lines) applied voltage (relative height of section B-C is exaggerated for purpose of illustration.).
In Capillary Force Actuators the electric potential is used to change the bridge shape and thereby the capillary pressure inside the bridge. In Figure 2.5, the profile of a capillary bridge with (solid lines) and without (dashed lines) applied voltage is illustrated, indicating the native contact angle of the liquid on the surface ($\theta_0$) and the apparent contact angle ($\theta_a$).

As discussed earlier, the electrostatic pressure is only significant on the liquid-gas surface in a region very close to contact line (within section B-C in Fig. 2.5). Over the rest of the surface, only the capillary pressure, $P_c$, is significant and the balance between the capillary pressure and the surface tension dictates the profile shape as captured by the Young-Laplace equation:

$$P_c = \sigma_{gl}\left(\frac{1}{r_1} + \frac{1}{r_2}\right)$$

(2.33)

Therefore, the profile shape over section $B-B'$ is within the Plateau sequence of shapes, the apparent contact angle serving as the bridge contact angle for this purpose. The local change in bridge shape in the vicinity of contact line is not considered. Given bridge volume, height, and apparent contact angle, the bridge shape and the capillary pressure inside may be determined. The capillary force may then be computed as the sum of capillary pressure and the surface tension contributions at any cross section along the profile.

Here, we consider three different cross sections along the profile and consider the force components to explore the equivalence between the three. Cross section C is chosen at the contact line (with bridge radius $r_c = r_c$ and contact angle $\theta_0$), cross section B is where
the electrostatic pressure loses its significance (with bridge radius $r_B$ and tangent angle $\theta$) and cross section A is at the neck of the bridge (with bridge radius $r_A = r_0$ and 90° tangent angle). The capillary force at these cross sections may be calculated as:

$$F_c^A = F_{\sigma}^A + F_p^A = -2\sigma_{gl}\pi r_0 \sin(90°) + \pi r_0^2 P_c$$

$$F_c^B = F_{\sigma}^B + F_p^B = -2\sigma_{gl}\pi r_B \sin(\theta_v) + \pi r_B^2 P_c$$

$$F_c^C = F_{\sigma}^C + F_p^C = -2\sigma_{gl}\pi r_c \sin(\theta_0) + \pi r_c^2 P_c$$

Since there is no net force acting upon the liquid slug in section A-B, we have:

$$F_c^A = F_c^B$$

from the balance of the forces. However, this is not the case for the liquid slug in section B-C because of the presence of electrostatic pressure along the profile which does not have a zero net force contribution (see Fig. 2.6).

Figure 2.6: Illustration of the forces exerted on the liquid slug in section B-C

Considering that the capillary pressure is compensated by the surface tension of the liquid, balance of force density along the profile in the radial direction may be written as:

$$-\sigma_{gl} \cos \theta_0 + \sigma_{gl} \cos \theta_v = \vec{F}_E^r$$

(2.38)
Kang [32] analytically calculated that the radial component of the electrostatic force density was:

\[ \tilde{F}_{E}^r = \frac{\varepsilon_d}{2t_d} v_d^2 \]  

(2.39)

Substitution of this into Eqn. (2.38) yields the Young-Lippmann equation:

\[ \sigma_{gl} (\cos \theta_v - \cos \theta_o) = \frac{\varepsilon_d}{2t_d} v_d^2 \]  

(2.40)

For the liquid slug in B-C, the balance of forces in the axial direction yields:

\[ 2\pi \sigma_{gl} \left[ \sin \theta_o r_C - \sin \theta_v r_B \right] - \pi P_c \left[ r_C^2 - r_B^2 \right] = F_E^z \]  

(2.41)

The liquid slug in section B-C where tangent angle varies from the native contact angle \( \theta_0 \) (at C) to the apparent contact angle \( \theta_v \) has a very small height (on the order of the dielectric thickness) and therefore:

\[ r_B \approx r_C \]  

(2.42)

Thus the total axial electrostatic force on the liquid slug may be found as:

\[ F_E^z = 2\pi \sigma_{gl} r_C \left[ \sin \theta_o - \sin \theta_v \right] \]  

(2.44)

A balance of axial forces on the slug yields

\[ F_E^z = F_C^B - F_C^C \]  

(2.45)

It is clear that if \( F_C^C \) was the only force exerted on the plate the force produced by CFA (\( F_C^C \)) would not be equal to the capillary force at the center of the bridge (\( F_C^A \)), unlike the situation for a bridge with no applied electric field where \( F_C^C = F_C^A \). However, if one considers the forces acting upon the box shown in Figure 2.7, which contains one of the plates and the half of the bridge adjacent to the plate, a balance of the forces in
equilibrium state requires that the capillary force at the center of the bridge $F_c^A$ be equal to the force exerted on the plate $F_c^C$.

Clearly, for the forces to balance there must be another force, acting upon the plate in addition to the capillary force $F_c^C$, which we will denote $F_E^C$. To balance, this force must be equal to the difference between the capillary force at the neck ($F_c^A$) –equivalently at cross section B ($F_c^B$)- and that acting upon the plate $F_c^C$:

$$F_E^C = F_c^A - F_c^B$$  \hspace{1cm} (2.46)

The origin of this force is claimed here to be the electrostatic force acting on the solid-gas interface of the plate in the vicinity of the contact line, this generated by the concentration of field on this surface. Therefore the total force on the plate would be the sum of the force exerted by the capillary bridge, $F_c^C$, and the force caused by electrostatic field, $F_E^C$, which is equal to the force at the center of the bridge:

$$F_c = F_c^C + F_E^C = F_c^C + (F_c^A - F_c^B) = F_c^B = F_c^A$$  \hspace{1cm} (2.47)

A numerical evidence of this claim is included in Appendix G.
An energy minimization approach may be used to find the apparent contact angle under electric field. The analysis employed uses the same assumptions as used in the treatment of sessile drops by Shapiro et al. [14], namely:

1) The bridge shape is axisymmetric and among the Plateau sequence of shapes. That is, gravitational body force has negligible effect, surface properties are axisymmetric, and plates are parallel.

2) The bridge is in static equilibrium.

3) Negligible surface roughness or heterogeneity exists; contact angle hysteresis may be ignored.

4) The drop is confined in a closed environment and evaporation losses are negligible.

To find the contact angle when electric potential is applied, we employ energy minimization. In addition to the surface energies ($E_{gl}$, $E_{ls}$) that were present in the no-voltage case, the total energy includes the capacitive potential energy of the electrostatic field, $E_{cap}$:

$$E = E_{gl} + E_{ls} + E_{cap}$$

For this analysis the electric field will be assumed to be uniform in the thin dielectric layer. Later, field non-uniformity will be examined in detail. With this assumption, the energy stored within, $E_d$, may be expressed as

$$E_d = (\frac{1}{2} \varepsilon_d E^2) V_d$$
where \( V_d \) is the volume of the dielectric layer penetrated by electric field, \( E \) is the electric field strength, and \( \varepsilon_d \) is the layer’s absolute permittivity. The volume \( V_d \) is that lying directly between the capillary bridge and the electrode:

\[
V_d = 2S_{ls}t_d
\]

where \( t_d \) is the thickness of the layer and the factor 2 arises from having the dielectric covering both electrodes. The field uniform strength in the dielectric layer is given by:

\[
E = \frac{v_d}{t_d}
\]

where \( v_d \) is the electric potential across the dielectric layer. Assuming the liquid bridge has electrical resistivity, this potential can be determined from voltage applied to the device, \( v \), and the resistances of the dielectric layer and the liquid bridge, \( R_d \) and \( R_l \):

\[
v_d = \frac{R_d}{2R_d + R_l}v
\]

Substitution of Eqns. (2.50)-(2.52) into Eqn. (2.49) yields an expression for total energy in the dielectric:

\[
E_d = \frac{\varepsilon_dS_{ls}}{4t_d}v^2\left(\frac{1}{1 + \frac{R_l}{R_d}}\right)^2
\]

Note that:

1) Other potential energy sources (e.g., those associated with ion layer capacitance) are very small and are considered negligible.

2) Changes in the ratio \( R_l/R_d \) with electrowetting was associated in [14] with contact angle saturation. With proper choice of materials \( R_l/R_d << 1 \). Unless
specified otherwise, it will be assumed that this is the case in the sequel and therefore \( v_d = v/2 \).

The potential energy stored in the external charging source is negative and equal in magnitude to twice that stored in the dielectric, as discussed in [33, 14]. As a result, the capacitive contribution to the potential energy is

\[
E_{\text{cap}} = E_d - 2E_d = -E_d \quad (2.54)
\]

Using Eqns. (2.48), (2.53), (2.54), the total energy stored in the device and its charging source is given by:

\[
E = \sigma_{gl}(S_{gl} - 2S_{ls}\cos(\theta_0)) - \frac{1}{4} \epsilon_{d} \frac{S_{ls}}{t_d} v^2 \quad (2.55)
\]

This expression may be rewritten as:

\[
E = S_{gl}\sigma_{gl} - 2S_{ls}\sigma_{gl}\cos(\theta_v) \quad (2.56)
\]

with

\[
\cos(\theta_v) = \cos(\theta_0) + \frac{\epsilon_{d}}{2t_d \sigma_{gl}} \left(\frac{v}{2}\right)^2 \quad (2.57)
\]

Since the shape of the bridge under electrical potential will remain within the Plateau sequence of the shapes, the contact angle achieved with electric field will be the angle, \( \theta_v \), given in Eqn. (2.57). This relation is the well-known Lippmann-Young equation [25].

### 2.4.2 Force Exerted

The force exerted by a bridge with an applied electric potential may be determined using Eqns. (2.57) and (2.30). First, Lippmann-Young, Eqn. (2.57), is used to determine the
contact angle. Then, the force may be found using Eqns. (2.18), (2.11), and (2.30), or alternately using Eqns. (2.25)-(2.27). Thus, the determination of force with electrical potential is the same as that without; only the contact angle needs to be modified in the analysis, as dictated by Lippmann-Young.

**An Example:** We consider an example bridge to illustrate the magnitude of forces generated. The numerical values used herein are motivated by figures provided in studies of EWOD droplet transport devices [21]. The bridge considered has a volume of $7.85 \times 10^7 \mu m^3$, equal to that of a cylindrical column with a 500 $\mu m$ radius and a 100 $\mu m$ height. For this study the volume will be kept constant and the height (consequently aspect ratio) and contact angle will be varied. The range of height examined is 20 $\mu m$ to 100 $\mu m$ (aspect ratio from 0.00895 to 0.1). The range of contact angle examined is 45° to 120° ($\pi/4$ to $2\pi/3$), which result in range of $\cos(\theta_c)$ from -0.5 to 0.707. Figure 2.8 shows the result, a plot of force exerted as a function of contact angle ($\cos(\theta_c)$) and the aspect ratio, $a$. Note that much greater forces and force variations with $\cos(\theta_c)$ are achieved when the aspect ratio is small.
2.4.3 Design Charts

In this section design charts are provided for the standard configuration. These may be used to ease the selection process of material properties and configuration dimensions. To make the charts universal, aspect ratio is chosen as one of the chart parameters:

\[ a = \sqrt{\frac{\pi h^3}{4V}} \approx \frac{h}{D} \]  

(2.58)

We also define two additional chart parameters:

\[ T = \{ t_d \ \text{in nanometers} \} \frac{\sigma_r}{\varepsilon_r} \]  

(2.59)

\[ B = \{ e_p \ \text{in MV/cm} \} \frac{\varepsilon_r}{\sigma_r} \]  

(2.60)

where \( \sigma_r \) denotes the surface tension of the liquid/gas interface relative to that of water with air (\( \sigma_w = 0.0728 \) N/m) i.e. \( \sigma_r = \frac{\sigma_{gl}}{\sigma_w} \).
Design must employ two charts, one relating force to $\cos(\theta_c)$ (see Fig. 2.9a), and another relating change in contact angle cosine, $n$, to voltage across the dielectric layer (see Fig. 2.9b). For universality, the force is written as $\frac{F_c}{\sigma h}$ where $F_c$ and $h$ are in units of $\mu N$ and $\mu m$, respectively (equivalently $N$ and $m$). Note that the force between two (effectively) hydrophobic surfaces ($\cos(\theta_c) < 0$) would be repulsive (positive force on the chart) except when the contact angle is close to $90^\circ$. For (effectively) hydrophilic surfaces ($\cos(\theta_c) > 0$), the force will always be attractive (negative force on the chart).

The second chart is simply a graph of the Lippmann-Young relation plotted for convenient units:

$$n = \cos(\theta_c) - \cos(\theta_0) = \frac{\varepsilon_d}{2t_d \sigma_{gl}} v_d^2 = \frac{\varepsilon_0 \times 10^9}{2 \sigma_w T} v_d^2$$

(2.61)

Breakdown voltage may be computed as the product of $B$ and $T$ with a correction factor for units:

$$v = v_b = t_d e_b = 0.1 \quad T \quad B$$

(2.62)

Also appearing on the second chart are curves of constant breakdown parameter $B$ computed via substitution of Eqn. (2.62) into (2.61) (to eliminate $T$ in the equation):

$$n = \frac{\varepsilon_0 \times 10^8 B}{2 \sigma_w} v_d$$

(2.63)
Figure 2.9: Design charts for standard configuration: (a) capillary force in terms of contact angle cosine, (b) change in contact angle cosine as a function of voltage
Example Design:

For a detailed explanation of the use of these design charts, we consider an example. An actuator with a maximum attracting force of $10 \, mN$ and stroke of $20 \, \mu m$ is desired for a specific MEMS application. Therefore, the bridge height should be chosen about $60 \, \mu m$ (so that the stroke needed is about $1/3$ of the bridge height). It is desired that the device fit within an area of $5mm \times 5mm$, therefore we choose the bridge diameter as $3mm$. The approximate aspect ratio of the device may be computed as $a = \frac{60 \, \mu m}{3 \, mm} = 0.02$. Consider the liquid and dielectric material are chosen to be water and silicon dioxide, respectively (i.e., $\sigma_r = 1, \varepsilon_r = 3.9$). Also suppose that the hydrophobic coating on top of the dielectric is very thin and has native contact angle $\theta_0 = 110^\circ$. The force related quantity in the first design chart may be computed as $\frac{F}{\sigma_r h} = \frac{10 \, mN}{1 \times 60 \, \mu m} \approx 167$.

From Fig. 2.9a, one may find the contact angle cosine needed to generate the desired force to be $\cos(\theta_r) = 0.6$. Therefore, the change in contact angle cosine may be computed as:

$$n = \cos(\theta_r) - \cos(\theta_0) = 0.6 - \cos(110^\circ) = 0.94$$

Suppose that the breakdown field strength of the dielectric material chosen has been determined via experiments to be: $e_b = 5 \, MV/cm$. Thus, the chart parameter $B$ may be computed as:

$$B = \frac{e_b \, \text{in} \, MV/cm \times \varepsilon_r}{\sigma_r} = 5 \times \frac{3.9}{1} = 19.5$$
As it may be observed from Fig 2.9b, for the desired force dielectric breakdown will occur before observing the desired change in contact angle cosine if $T < 4$ (approximately). Therefore the dielectric thickness may be chosen based on the voltage desirable to be applied to the system. For the lowest positive value, $v_d \approx 7.5 \text{ volts}$, we find $T = 4$, which corresponds to dielectric thickness of:

$$t_d \text{ in nanometers} = T \frac{\varepsilon_r}{\sigma_r} = 4 \times \frac{3.9}{1} = 15.6 \text{ nm}$$

As it may be observed from Fig 2.9a, higher levels of nondimensional force are achieved with lower aspect ratio bridges. This has consequences for CFA operation. Since the bridge has constant volume, the aspect ratio will decrease as the plates move closer together. Therefore, when sufficient voltage is applied and attractive forces are generated, the aspect ratio will decrease as the plates are drawn together. This means moving to a lower aspect ratio curve in Fig. 2.9a during operation and therefore the production of greater capillary force. This situation is reversed when the surfaces become hydrophobic. In this case, the repulsive capillary force present may be used to push the plates farther from each other. Higher bridge height results in higher aspect ratio and consequently an attenuation of the repulsive capillary force. Similar behavior is expected from the alternative configurations discussed later in Chapter 5. The only exception would be the application of CFA to a flexible membrane, in which case the bridge height remains constant and therefore only the initial aspect ratio will determine the change in capillary force while the device is in operation.
3 Actuator Analysis

3.1 Analytical Approximation of the Force and Pressure

Complex bridge profiles such as the unduloid and nodoid, which are described by elliptic integrals, prevent a direct analytical investigation of how force and pressure are related to design parameters. Semi-analytic investigations using the techniques of Chapter 2 have revealed that low aspect ratio bridges provide greater force than those with higher aspect ratios. Here, approximations of the bridge profile will be examined for the low aspect ratio case, which provide insight helpful to actuator design.

To recap, the force exerted by a liquid bridge with an electric potential (assuming \( R_i \ll R_d \)) is:

\[
F_c = -\frac{\partial}{\partial h} E = -\sigma_{gl} \frac{\partial S_{gl}}{\partial h} + 2\sigma_{gl} \cos(\theta_0) \frac{\partial S_{ls}}{\partial h} + \frac{1}{4} \frac{\varepsilon_d}{t_d} \nu^2 \frac{\partial S_{ls}}{\partial h}
\]  

(3.1)

Independent of contact angle, we may state that

\[
\frac{\partial S_{gl}}{\partial h} > 0 \quad \frac{\partial S_{ls}}{\partial h} < 0
\]  

(3.2)

As a result, the first and third terms of Eqn. (3.1) are always negative (i.e., attractive force between the plates). The second term is negative (i.e., attractive) only when \( \cos \theta_0 > 0 \) (i.e., \( \theta_0 < 90^\circ \)). While it is not explicit in (3.1), the first and second terms are functions of voltage as \( S_{gl} \) and \( S_{ls} \) are dependent on \( \theta_v \), which is dependent on \( \nu \). For low aspect ratio bridges, the dependence of \( \partial S_{gl} / \partial h \) on \( \theta_v \) (hence \( \nu \)) is much stronger than that of
\( \frac{\partial S_{gl}}{\partial h} \). In the case of a low aspect ratio cylindrical bridge \((a < 0.1)\) with contact angle near 90° (i.e., \(\theta_c \in [60^\circ, 120^\circ]\)) one may use a bridge approximation [see Appendix E]:

\[
S_{gl} = 2\pi rh \\
S_{ls} = \pi r^2 \\
V = \pi r^2 h
\]

where \(r\) is the (approximately constant) radius of the bridge. It may be demonstrated that:

\[
\left. \frac{\partial S_{gl}}{\partial h} \right|_{r_{fixed}} = \pi r \\
\left. \frac{\partial S_{ls}}{\partial h} \right|_{r_{fixed}} = -\pi r \left( \frac{r}{h} \right)
\]

Since \(\frac{r}{h} \gg 1\) for low aspect ratio bridges, \(\left| \frac{\partial S_{ls}}{\partial h} \right| > \left| \frac{\partial S_{gl}}{\partial h} \right|\).

Substitution into Eqn. (3.1) then yields a useful approximation:

\[
\tilde{F}_c = -\pi r \sigma_{gl} - 2\pi r^2 \frac{2\pi r^2}{h} \sigma_{gl} \cos(\theta_c) - \frac{\pi r^2 \varepsilon_{d}}{4ht_d} v^2
\]

or

\[
\tilde{F}_c = -2\pi r \sigma_{gl} \left( \frac{1}{2} + \frac{r}{h} \cos(\theta_c) \right)
\]

In Figure 3.1, this estimate of the force is compared to that calculated using the actual liquid bridge profile (Section 2.3.2). As the results show (Figure 3.2), the approximate relations are very accurate \((< 2\% \text{ error})\) for low aspect ratios.
Figure 3.1: Force estimate as a function of $\cos(\theta_v)$ and $a$

Figure 3.2: Relative error in force estimate as a function of $\cos(\theta_v)$ and $a$
The approximate force, Eqn. (3.8), is non-dimensionalized by $\sigma_{gl}V/h^2$:

$$\frac{\tilde{F}_c}{\sigma_{gl}V/h^2} = -\frac{h}{r} - 2\cos(\theta_o) - \frac{\varepsilon_d}{4t_d\sigma_{gl}}v^2$$

(3.10)

Note that the capacitive energy per unit wetted area of the device is

$$\frac{E_{cap}}{S_{gl}} = -\frac{\varepsilon_d}{4t_d}v^2$$

(3.11)

As this quantity has the same units as surface tension, define an effective capacitive surface tension as $\sigma_{cap} = E_{cap}/S_{gl}$. Thus, the nondimensional approximate force relation, Eqn. (3.10), may be rewritten as:

$$\frac{\tilde{F}_c}{\sigma_{gl}V/h^2} = -\frac{h}{r} - 2\cos(\theta_o) + \frac{\sigma_{cap}}{\sigma_{gl}}$$

(3.12)

In Figure 3.3, the nondimensionalized actual force $F_c/(\sigma_{gl}V/h^2)$ is plotted as a function of aspect ratio and contact angle, demonstrating its nearly multilinear dependence on these variables, as was indicated by the approximate relation, Eqn. (3.12).

Figure 3.3: Nondimensionalized actuator force, $F_c/(\sigma_{gl}V/h^2)$ as a function of $\cos(\theta_o)$ and $a$
Denote the three terms in Eqn. (3.8) as:

\[ F_1 = -\sigma_{gl} \pi r \] (3.13)
\[ F_2 = -\frac{2\pi r^2}{h} \sigma_{gl} \cos(\theta_0) \] (3.14)
\[ F_3 = -\frac{\pi r^2 \varepsilon_{d} v^2}{4 h t_d} \] (3.15)

For the case where \( \theta_0 \) is near 90°, the second term will be negligible. Consider the ratio of the first and third terms in Eqn. (3.8):

\[ \frac{F_3}{F_1} = \frac{1}{4} \left( \frac{\varepsilon_{d} v^2}{t_d \sigma_{gl}} \right) \left( \frac{r}{h} \right) \] (3.16)

From the Lippmann-Young equation (2.4) we have:

\[ \frac{\varepsilon_{d} v^2}{t_d \sigma_{gl}} = 8(\cos(\theta_v) - \cos(\theta_0)) \] (3.17)

From previous research with EWOD sessile droplets, we may expect the minimum contact angle \( \theta_v \) to occur near 45° due to contact angle saturation (\( \theta_v = \theta_{sat} \)) and that the no-voltage contact angle \( \theta_0 \) is near 110°. Using these values, we may state

\[ \frac{\varepsilon_{d} v_{sat}^2}{t_d \sigma_{gl}} \approx 7 \] (3.18)

and therefore

\[ \left[ \frac{F_3}{F_1} \right]_{\text{max}} \approx \frac{7}{4} \left( \frac{r}{h} \right) \] (3.19)

This indicates that if the bridge aspect ratio is small (i.e. \( r/h \gg 1 \)), the maximum value of the third term (that due to the change in capillary pressure) is significantly greater than that of the first term (that due to the surface tension acting along the contact line).
Therefore, a reasonable approximation of the force and pressure exerted by a low aspect ratio bridge (with $\theta_0$ near 90°) under significant applied voltage is:

$$\vec{F}_c \approx F_3 = -\frac{\varepsilon_d \pi r^2}{4t_d h} v^2 = -\frac{\varepsilon_d S_{hi}}{4t_d h} v^2$$

(3.20)

and so the pressure exerted is approximately

$$\vec{P} \approx -\frac{\varepsilon_d}{4t_d h} v^2$$

(3.21)

**Alternate Expressions**

Capillary force may be expressed in terms of any two of the following four configuration parameters: radius ($r$), height ($h$), aspect ratio ($a$) and volume ($V$). An expression in terms of nondimensional variable aspect ratio and volume enables us to employ the capillary force graphs (Figures 3.1-3.3) to compute the force for any given bridge dimension. As stated earlier, the capillary force may be expressed in terms of configuration parameters wetting radius and height as:

$$\vec{F}_c = -2\pi r \sigma_g \left( \frac{1}{2} + \frac{r}{h} \cos(\theta_r) \right)$$

(3.22)

Employing Eqn. (2.9), expressions for configuration parameters $r$ and $h$ in terms of aspect ratio and bridge volume may be derived:

$$h = \frac{3}{\sqrt{\pi}} \frac{4}{V} a^2$$

(3.23)

$$r = \frac{h}{2a} = \frac{3}{\sqrt{2\pi}} \frac{1}{V} a^{-1}$$

(3.24)

Substitution of these into Eqn. (3.22) yields:
\[ F_e = -\sigma_{vl} \sqrt{\frac{\pi^2 V}{2a}} \left( 1 + \frac{\cos(\theta_v)}{a} \right) \]  

(3.25)

Force may also be expressed in terms of bridge volume and height. The radius of the bridge may be found in terms of volume and height as:

\[ r = \sqrt{\frac{V}{\pi h}} \]  

(3.26)

Substitution then yields:

\[ F_e = -2\pi \sigma_{vl} \left( \frac{1}{2} \sqrt{\frac{V}{\pi h}} + \frac{V}{\pi h^2} \cos(\theta_v) \right) \]  

(3.27)

For low aspect ratios, the second term would be dominant. This confirms that force is linearly dependent on drop volume for a configuration with fixed spacing between two electrodes.

More generally, one may consider the contact angle and bridge volume as two independent manipulatable variables. From this viewpoint, the force may be controlled by varying either one of these variables. Since the cosine of the contact angle may more easily undergo a large percent change in value than the bridge volume, manipulation of the contact angle is a more effective means of actuation.

### 3.2 Limits to Actuation Force

Employing high voltages in microsystems is undesirable. The allowable voltage thus imposes a limit on the maximum force and pressure that may be achieved with capillary force actuation. Two other phenomena also constrain the force that may be achieved: (1)
EWOD contact angle saturation; and (2) electrical breakdown in the dielectric layer. The reader should note that throughout Sections 3.2 and 3.3 we will treat the attractive force as positive for simplicity of presentation.

To investigate the impact of contact angle saturation, we will rewrite Eqn. (3.17) so as to define the voltage at which contact angle saturation occurs, $v_{sat}$:

$$v_{sat}^2 = \frac{8t_d \sigma_{gl} n_{sat}}{\varepsilon_d}$$  \hspace{1cm} (3.28)

$$n_{sat} = \cos(\theta_{sat}) - \cos(\theta_0)$$  \hspace{1cm} (3.29)

where $n$ is dependent on many factors and may only be determined via experiment.

From previous EWOD results in the literature, $n_{sat} \in [0.7, 1.0]$.

The voltage limit set by electrical breakdown is linearly dependent on the thickness of the dielectric layer, where the constant of proportionality, $e_b$, is a material property known as the breakdown field strength. We may define the maximum voltage without breakdown (ignoring liquid resistance) as

$$v_b = 2e_b t_d$$  \hspace{1cm} (3.30)

where the factor 2 appears since each dielectric layer has a voltage drop of half the applied voltage.

Recall that the maximum force for low aspect ratio bridges was approximately

$$\tilde{F}_{max} = -\frac{\varepsilon_d S_h}{4t_d h} v_{max}^2$$  \hspace{1cm} (3.31)
Substitution of Eqns. (3.28) and (3.30) yield expressions for the limits on force imposed by contact angle saturation \((\text{sat})\) and electrical breakdown \((b)\):

\[
F_{\text{sat}} = 2n_{\text{sat}} \sigma_{gl} \frac{S_b}{h} \tag{3.32}
\]

\[
F_b = \varepsilon_d \varepsilon_b^2 t_d \frac{S_{ts}}{h} \tag{3.33}
\]

\[
|F| < \min(F_{\text{sat}}, F_b) \tag{3.34}
\]

Which of these two constraints governs the maximum force production depends on the thickness of the dielectric layer, the liquid/gas surface tension, the contact angle saturation, and the layer’s permittivity and breakdown field strength. From Eqns. (3.32) and (3.33), the constraints on the pressure exerted are:

\[
P_{\text{sat}} = \frac{2n_{\text{sat}} \sigma_{gl}}{h} \tag{3.35}
\]

\[
P_b = \frac{\varepsilon_d \varepsilon_b^2 t_d}{h} \tag{3.36}
\]

Let us consider how the constraints on force (and consequently pressure) vary with dielectric layer thickness, as illustrated in Figure 3.4.
As indicated in the figure, below a critical layer thickness \( t_d^* \) the maximum force is dictated by dielectric breakdown. Above \( t_d^* \) contact angle saturation dictates the force constraint. The critical thickness value may be found by equating the two constraints:

\[
t_d^* = \frac{2n\text{sat}\sigma_{gl}}{\epsilon_0\epsilon_r\epsilon_b^2}
\]  

(3.37)

Note that \( t_d^* \) is independent of actuator size. Insight is provided by consideration of some typical ranges of values:

\[
\begin{align*}
\epsilon_0 &= 8.85 \times 10^{-12} \text{ F m}^{-1} \\
\epsilon_r &\in [3, 50] \\
n &\in [0.7, 1.0] \\
\sigma_{gl} &\in [0.04, 0.5] \text{ N m}^{-1} \\
\epsilon_b &\in [10^7, 10^9] \text{ V m}^{-1}
\end{align*}
\]

Therefore, the critical thickness value will be in the range of 0.1 nm to 30 \( \mu \text{m} \). Of course, the assumptions made in analysis would not be accurate for the lower end of this range (i.e. less than 1 nm). Consider now the voltages necessary to achieve these limits on force, \( v_{\text{sat}} \) and \( v_b \). The dependence of these values on the layer thickness is depicted in Figure 3.5. This result, in combination with the previous figure demonstrates the desirability of selecting the dielectric layer thickness to be equal to the critical value. If thickness \( t_d \) is chosen to be less than \( t_d^* \), the full force capability of the bridge (\( F_{\text{sat}} \)) will not be realized as dielectric breakdown will occur first. If it is chosen to be greater than \( t_d^* \), the device will require a greater voltage than necessary to achieve \( F_{\text{sat}} \).
If the actuator is manufactured such that the dielectric layer thickness, \( t_d \), is equal to \( t_d^* \), then the voltage required to achieve the maximum force \( F_{sat} \) is \( v^* = 2e_b t_d^* \), or

\[
v^* = \frac{4n_{sat}\sigma_{gl}}{\varepsilon_0\varepsilon_r e_b}
\]  

(3.38)

In this case, both the maximum force and the required voltage to achieve it vary linearly with \( \sigma_{gl} \). This allows us to determine a constraint on performance, independent of the liquid chosen. Suppose that manufacturing will achieve \( t_d = t_d^* \) for whatever liquid that is selected. Solving Eqn. (3.38) for \( \sigma_{gl} \) and substituting into Eqn. (3.35) yields:

\[
\frac{P_{max}}{v^*} < \frac{\varepsilon_0\varepsilon_r e_b}{2h}
\]  

(3.39)

Therefore, once the dielectric material is chosen (\( \varepsilon_r, e_b \)) and the bridge height specified, a constraint on the ratio of exerted pressure to voltage is dictated, independent of the liquid chosen or dielectric layer thickness. This is illustrated in Figure 3.6.
Consider a 50 \( \mu m \) high capillary bridge of water (with low aspect ratio) \((n=1, \sigma = 70 \times 10^{-3} N/m, h = 5 \times 10^{-5} m)\). The maximum pressure dictated by saturation is:

\[
P_{sat} = 2800 \frac{N}{m^3} = 0.4 psi.
\]

For a silicon dioxide dielectric layer \((e_b=5 \times 10^8 V/m, \varepsilon_r = 3.9)\) the critical thickness would be \(t_d^* = 21 \text{ nm}\). If the device was designed with this layer
thickness, the electric potential needed to achieve the maximum force would be $v^* = 16.2$ volts. If silicon nitride were chosen ($e_r = 6$, $\varepsilon_r = 6$), the critical thickness would be $t_d^* = 3$ nm and 5.3 volts would be required to achieve the maximum force. (The question of whether such thin layers may be manufactured and whether their electrical properties are consistent with the data presented is not considered here.)

### 3.3 Ralston’s Electrowetting Limit

In recent papers by Peykov et al. [17] and Quinn et al. [18] a hypothesis is presented regarding contact angle saturation which may have important implications for CFA. For a low aspect ratio bridge, the pressure exerted with contact angle $\theta_0$ is:

$$P_c = \sigma_{gl} \left[ \frac{1}{r_{cyl}} \frac{2}{h} \cos(\theta_0) \right]$$

(3.40)

In which the first term is due to the change in energy of the liquid/gas interface with a virtual displacement and the second term is due to the change in energy of the solid/liquid and solid/gas interfaces. From Young’s equation we have:

$$\cos(\theta_0) = \frac{\sigma_{gl} - \sigma_{ls}}{\sigma_{gl}}$$

(3.41)

Substitution yields:

$$P_c = \frac{\sigma_{gl}}{r_{cyl}} \frac{2[\sigma_{gl} - \sigma_{ls}]}{h}$$

(3.42)

When contact angle changes, the energy change is due to the difference between solid/gas and solid/liquid surface tension. Peykov et al. [17] and Quinn et al. [18] point out that, in
EWOD the solid/liquid surface tension is effectively changed. It is hypothesized that this surface tension cannot be made negative due to a thermodynamic argument, ie:

$$\sigma_{ls}^{\text{eff}} \geq 0$$

Therefore, according to this hypothesis, the lowest (most negative) pressure that can be achieved is:

$$P_c = -\frac{\sigma_{gl}}{r_{cyl}} - \frac{2\sigma_{gs}}{h}$$  \hspace{1cm} (3.43)

And the maximum change in pressure is:

$$\Delta P_c = -\frac{2\sigma_{gs}}{h}$$  \hspace{1cm} (3.44)

This is illustrated in Figure 3.8.

In the previous section, contact angle saturation was treated by parameter $n_{\text{sat}} = \cos(\theta_{\text{sat}}) - \cos(\theta_0)$. In terms of Ralston’s hypothesis:

$$n_{\text{sat}} = \cos(\theta_{\text{sat}}) - \cos(\theta_0) = \left[ \frac{\sigma_{gs}}{\sigma_{gl}} \right] - \left[ \frac{\sigma_{gs} - \sigma_{ls}}{\sigma_{gl}} \right] = \frac{\sigma_{ls}}{\sigma_{gl}}$$  \hspace{1cm} (3.45)

Substitution into Eqn. (3.35) yields:

$$P_{\text{sat}} = \frac{2n_{\text{sat}} \sigma_{gl}}{h} = \frac{2\sigma_{ls}}{h}$$  \hspace{1cm} (3.46)
Technically, this corresponds to the change in pressure exerted within applied voltage, not to the absolute value of the pressure exerted. As it employs the previous result where force contributions $F_1$ and $F_2$ were considered negligible. Recalling that maximum pressure at electrical breakdown employed the same assumptions, we may equate the pressures:

$$p_{\text{sat}} = \frac{2\sigma_{ls}}{h} \quad P_b = \frac{\varepsilon_d \varepsilon_r^2 t_d}{h}$$

and find the critical layer thickness as:

$$t_d^* = \frac{2\sigma_{ls}}{\varepsilon_d \varepsilon_r \varepsilon_b}$$

All the conclusions presented earlier regarding $v^*$ and $P/v^*$ hold.

### 3.4 Design Optimization

To facilitate the design of parallel plate CFA considering the constraints imposed by allowable device voltage, electrical breakdown, and contact angle saturation, we will develop approximate relations between the performance metrics / constraints and the design parameters. For performance, we will consider maximum force, $F_{\text{max}}$, maximum change in force, $\Delta F_{\text{max}}$, maximum pressure exerted, $P_{\text{max}}$, and maximum force density, $f_{\text{max}}$ (ratio of force to volume). For a low aspect ratio bridge, the force at maximum voltage (assuming neither breakdown nor contact angle saturation) is:

$$F_{\text{max}} = -\pi r \sigma_{s_l} - 2\pi r \left(\frac{r}{h}\right) \sigma_{s_l} \cos(\theta_b) - \frac{\pi}{4} r \left(\frac{r}{h}\right) \frac{\varepsilon_d}{t_d} v_{\text{max}}^2$$

(3.48)
As discussed earlier, only the third term will change significantly with the application of electric field, so the maximum change in force is:

$$\Delta F_{\text{max}} = -\frac{\pi}{4} r \left( \frac{r}{h} \right) \frac{\varepsilon_d}{t_d} \nu_{\text{max}}^2$$

(3.49)

The maximum pressure exerted is $F_{\text{max}}$ divided by area $A$:

$$P_{\text{max}} = -\frac{1}{r} \sigma_{g/l} - 2 \left( \frac{1}{h} \right) \sigma_{g/l} \cos(\theta_0) - \frac{1}{4} \frac{1}{h^2} \frac{\varepsilon_d}{t_d} \nu_{\text{max}}^2$$

(3.50)

The maximum force density is defined as the maximum force divided by the volume of the capillary bridge $V$:

$$f_{\text{max}} = -\left( \frac{1}{h} \right) \frac{1}{r} \sigma_{g/l} - 2 \left( \frac{1}{h^2} \right) \sigma_{g/l} \cos(\theta_0) - \frac{1}{4} \left( \frac{1}{h^2} \right) \frac{\varepsilon_d}{t_d} \nu_{\text{max}}^2$$

(3.51)

Table 3.1-3.3 show the dependence of these performance metrics upon the design parameters in three cases: (1) maximum voltage dictated by voltage supply and less than those values prescribed by dielectric breakdown or contact angle saturation; (2) maximum voltage constrained by dielectric breakdown; and (3) maximum voltage dictated by contact angle saturation. In the table $x^k$ indicates that the performance metric is dependent on the parameter to the $k^{th}$ power. The notation $x^k + x^j$ indicates that the metric is dependent on the parameter in a mixed fashion, with some terms having a power dependence of $k$ and others a power dependence of $j$. 
Table 3.1: Dependence of performance metrics on design parameters, $v_{\text{max}} < \min(v_b, v_{\text{sur}})$

<table>
<thead>
<tr>
<th>Metric</th>
<th>Equation</th>
<th>Parameter Dependence</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_{\text{max}}$</td>
<td>$-\pi r {\sigma}<em>{gl} - \frac{2 \pi r^2}{h} {\sigma}</em>{gl} \cos(\theta_0) - \frac{\pi r^2}{4h} \varepsilon_d t_d v_{\text{max}}^2$</td>
<td>$x^0 + x^1$ $x^0 + x^1$ $x^0 + x^1$</td>
</tr>
<tr>
<td>$\Delta F_{\text{max}}$</td>
<td>$-\frac{\pi r^2}{4h} \varepsilon_d v_{\text{max}}^2$</td>
<td>$x^0$ $x^0$ $x^1$</td>
</tr>
<tr>
<td>$P_{\text{max}}$</td>
<td>$-\frac{1}{r} {\sigma}<em>{gl} - 2 \left( \frac{1}{h} \right) {\sigma}</em>{gl} \cos(\theta_0) - \frac{1}{4} \left( \frac{1}{h} \right) \varepsilon_d t_d v_{\text{max}}^2$</td>
<td>$x^0 + x^1$ $x^0 + x^1$ $x^0 + x^1$</td>
</tr>
<tr>
<td>$f_{\text{max}}$</td>
<td>$-\frac{1}{rh} {\sigma}<em>{gl} - \frac{2}{h^2} {\sigma}</em>{gl} \cos(\theta_0) - \frac{1}{4h^2} \varepsilon_d t_d v_{\text{max}}^2$</td>
<td>$x^0 + x^1$ $x^0 + x^1$ $x^0 + x^1$</td>
</tr>
</tbody>
</table>

Table 3.2: Dependence of performance metrics on design parameters, $v = v_b = 2 e_b t_d$

<table>
<thead>
<tr>
<th>Metric</th>
<th>Equation</th>
<th>Parameter Dependence</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_{\text{max}}$</td>
<td>$-\pi r {\sigma}<em>{gl} - \frac{2 \pi r^2}{h} {\sigma}</em>{gl} \cos(\theta_0) - \pi \varepsilon_d e_b^2 t_d r^2 h$</td>
<td>$x^0 + x^1$ $x^0 + x^1$ $x^0 + x^1$</td>
</tr>
<tr>
<td>$\Delta F_{\text{max}}$</td>
<td>$-\pi \varepsilon_d e_b^2 r^2 t_d h$</td>
<td>$x^0$ $x^0$ $x^0 + x^1$</td>
</tr>
<tr>
<td>$P_{\text{max}}$</td>
<td>$-\frac{1}{r} {\sigma}<em>{gl} - 2 \frac{1}{h} {\sigma}</em>{gl} \cos(\theta_0) - \varepsilon_d e_b^2 t_d h$</td>
<td>$x^0 + x^1$ $x^0 + x^1$ $x^0 + x^1$</td>
</tr>
<tr>
<td>$f_{\text{max}}$</td>
<td>$-\frac{1}{rh} {\sigma}<em>{gl} - \frac{2}{h^2} {\sigma}</em>{gl} \cos(\theta_0) - \varepsilon_d e_b^2 t_d h^2$</td>
<td>$x^0 + x^1$ $x^0 + x^1$ $x^0 + x^1$</td>
</tr>
</tbody>
</table>
Table 3.3: Dependence of performance metrics on design parameters, \( v = \nu_{sat} = \sqrt{\frac{8n t_d \sigma_{gl}}{\varepsilon_d}} \)

<table>
<thead>
<tr>
<th>Metric</th>
<th>Equation</th>
<th>Parameter Dependence</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>( \sigma_{gl} )</td>
</tr>
<tr>
<td>( F_{max} )</td>
<td>(-\pi r \sigma_{gl} - \frac{2\pi r^2}{h} \sigma_{gl} \cos(\theta_0) - \frac{2\pi r^2}{h} n \sigma_{gl} )</td>
<td>( x^1 )</td>
</tr>
<tr>
<td>( \Delta F_{max} )</td>
<td>(-\frac{2\pi r^2}{h} n \sigma_{gl} )</td>
<td>( x^1 )</td>
</tr>
<tr>
<td>( P_{max} )</td>
<td>(-\frac{1}{r} \sigma_{gl} - \frac{2}{h} \sigma_{gl} \cos(\theta_0) - \frac{2}{h} n \sigma_{gl} )</td>
<td>( x^1 )</td>
</tr>
<tr>
<td>( f_{max} )</td>
<td>(-\frac{1}{rh} \sigma_{gl} - \frac{2}{h^2} \sigma_{gl} \cos(\theta_0) - \frac{2}{h^2} n \sigma_{gl} )</td>
<td>( x^1 )</td>
</tr>
</tbody>
</table>

### 3.5 Scaling Analysis

#### 3.5.1 Introduction

The effectiveness of any actuation technology is highly dependent on the actuator’s size. This is of particular importance at the microscale. The scaling of force production was first comprehensively examined by Trimmer [34, 35]. Later, Madou [3] examined scalings specifically for MEMS. In such analyses, the manner in which force varies with the linear dimension of the device, parameterized by the variable \( L \), is considered. For practical reasons, certain linear dimensions associated with small features of the actuator may be frozen during scaling analysis. In Table 3.4, we repeat Madou’s summary of scalings for the reader’s benefit.
Let us first consider parallel plate electrostatic actuators. The breakdown voltage is the limiting constraint on the maximum force. This depends on the electrode spacing, gas pressure, and type of the gas within. The dependence on spacing is not monotonic as the Paschen curve indicates [3]. A minimum breakdown voltage is realized at a certain value of plate separation, $h^*$, and higher breakdown voltages can be achieved for either larger or smaller clearances. The breakdown electric field for larger separation than $h^*$ is almost constant ($E^0$) while it scales as $L^{0.5}$ for smaller values. The force may be calculated from the electrostatic potential energy $E_{ex}$ via

$$F = -\frac{\partial E_{ex}}{\partial x}$$

(3.52)

$$E_{ex} = \frac{1}{2} \varepsilon_0 S_{\|} h E^2$$

(3.53)

where $x$ is a virtual displacement in the direction of force to be examined. The force achieved is then:

$$F = \frac{1}{2} \varepsilon_0 \frac{\partial}{\partial x} (S_{\|} h E^2)$$

(3.54)
When the gap between plates is greater than $h^*$, the force scales as $L^2$ since

$$E \propto L^0 \Rightarrow F \propto \frac{1}{L}L^3 = L^2$$

(3.55)

However, in the regime where $h < h^*$, the force scales as $L^0$ as we have:

$$E \propto L^{-0.5} \Rightarrow F \propto \frac{1}{L}(L^3 L^{-1}) = L$$

(3.56)

Similar scaling analysis has shown that electromagnetic force scales as $L^4$ for constant current density, $L^3$ for constant heat flow per unit area of the windings and $L^2$ for constant temperature difference between windings and environment [35].

### 3.5.2 Capillary Bridge Force

First, the scaling of the capillary force of a liquid bridge (without electric field) will be examined. We start by noting that surface tension forces scale as the first power of dimension [3, 35], a result which Trimmer referred to as “an absolutely delightful scaling”. For the reader’s benefit, we reintroduce Eqn. (2.30) governing the capillary force:

$$F_c = -2\pi r_0 \sigma_p (1 - p)$$

(3.57)

Employing Eqns (2.11), (2.12) and (2.16) to substitute an expression in terms of $p$ and $\theta_0$ for $r_0$, yields:

$$F_c = -\pi h \sigma_p \frac{(1 - p) |p|}{\zeta(P(\theta_0, p), p)}$$

(3.58)
Since nondimensional pressure $p$ and contact angle $\theta_0$ are independent of scaling, the capillary force (including both capillary pressure and surface tension contributions) is linearly dependent on the dimension of the device (i.e., scales as $L$).

### 3.5.3 Capillary Force Actuation

For the capillary force actuator considered, the scaling depends on the dielectric layer thickness ($t_d$) used. Recall that the critical thickness $t_d^*$ is dependent on material properties alone and is independent of actuator size. If $t_d < t_d^*$, dielectric breakdown will constrain the maximum force produced

$$F_p = \varepsilon e^2 t_d \frac{S_{ls}}{h} \quad (3.59)$$

and the force will scale as $L$ ($t_d$ is fixed in the scaling analysis). Consider now the case where $t_d^* < t_d$. Contact angle saturation during electrowetting will be the governing constraint on maximum force:

$$F_{sat} = 2 n_{sat} \sigma_{gl} \frac{S_{ls}}{h} \quad (3.60)$$

Clearly, the maximum force in this case scales as $L$ as well. Note that this analysis employed the simplified force equations of Section 3.1 which neglected the force contributions of surface tension acting at the contact line and the zero-voltage capillary pressure (respectively, $F_1$ and $F_2$ in Eqn. (3.14) and (3.15)). However, those terms scale as $L$ as well, as discussed in Section 3.4.2. Therefore, the force produced by a capillary force actuator scales linearly with actuator dimension.
3.6 Numerical Approaches to Capillary Bridges

As well as analysis, numerical approaches will be of interest for the examination of the device. Numerical methods can be easily generalized for more complex configurations and enable us to verify our analytically based results. In our case, using numerical methods will also guide us to new ways of checking the device stability.

For determining the bridge shape and capillary force numerically, a simplified version of the algorithm employed in Surface Evolver software is used. Surface Evolver [36] is a program that iteratively calculates a three-dimensional surface subject to constraints so as to determine the minimum energy surface. The main idea of the code is to approximate the axisymmetric profile of the bridge by a set of vertices connected by straight edges and then an iterative process to minimize the total energy while holding the volume, height, and contact angles constant. The reason to write a special code rather than using the existing software is to speed up the calculations for the sequence of boundary conditions with small changes in each step which will allow us to use the final profile of the former step as the beginning profile of the new step. This will also give us more power in controlling the desired accuracy and tuning the coefficients used in the algorithm.

In every step, the final shape of bridge can be approximately achieved using the iterative method for any set of device parameters including applied voltage, distance between plates, dielectric constant of the plates and their thickness. Having the amount of energy for the equilibrium shape for different values of \( h \), one can use interpolation for the total energy as a function of the distance between plates. The derivative of this function with respect to \( h \) is the capillary force function in terms of the distance between plates and all
other device parameters. A detailed explanation of the numerical code employed is included in Appendix H.

Figure 3.9: Numerical approximation of axisymmetric capillary bridge in Matlab code

Surface Evolver:

An alternative approach for calculating the capillary force provided by the device is to approximate the droplet shape and find the amount of force using geometrical quantities provided by employing Surface Evolver program. Surface Evolver is a program that minimizes the energy of given three-dimensional surface subject to constraints by evolving it down the energy gradient. The surface is represented as a union of simplices and an iterative finite element method is used. In each iteration step, the gradient of the energy with respect to coordinates of each non-fixed vertex is calculated. This is referred to as the “force” acting at that vertex. Projecting the force of every vertex on to the tangent space of the constraints allows the next position of the vertices to be obtained. This step also includes finding a constraint correction that compensates the prior error in the constraint. Other than the energy minimizing iteration step, other basic operations such as refinement, equiangu

lation, and vertex averaging have been defined to optimize
the final result. As with any other iteration program, one might face numerical difficulties using the program for complicated configurations and unavoidable large number of iterations to get to desired approximation of final result. Using this program enables us to find the minimum-energy shape of liquid droplets in complex geometrical configurations for which analytical approaches would fail. One must be careful to avoid an inappropriate order of operation steps which can result in one of the unstable perturbations of the stable solution. The figure below shows the final result of a sample parallel plate configuration.

Figure 3.10: Final result of Surface Evolver for a sample parallel plate configuration

The initial shape is defined to be a cylinder between two plates. This is done by introducing 20 equally spaced vertices in a circle on each plate. Increasing the number of vertices results in faster convergence and greater accuracy. After specifying the coordinates of the vertices in the code, a list of edges with two vertices for each is declared and facets composed of closed paths of edges are defined. Because the process of defining all the geometrical specifications is time consuming a simple C++ code was developed to generate Surface Evolver definitions for axisymmetric configurations automatically.
Constraints considered for this configuration consist of fixed height for the initial vertices on the two plates and fixed volume. The contact angle constraints are handled by giving an energy integrand to the vertices on the plates. This specification results in the same final energy level as the contact angle.

Surface Evolver offers means to define and calculate variables (called “method”) in terms of the element properties for individual elements. In addition, sum of these variables (called “quantity”) for a particular set of elements (usually with specific characteristic) may be calculated and used. Here we will employ this to calculate the perimeter $L_{SE}$ and area $S_{SE}$ of the contact surface of the drop.

The pressure is also available as the Lagrange multiplier of the volume constraint. The value extracted from the software may be expressed in terms of defined parameters in this text as:

$$p_{c}^{SE} = \frac{P_{c}}{\sigma_{gl}} = \frac{2P}{r_{0}}$$

The force may be calculated as the sum of contributions from capillary pressure and surface tension at the contact line:

$$F_{c}^{SE} = F_{p}^{SE} + F_{\sigma}^{SE} = [S_{h}^{SE} P_{c}^{SE} - L_{SE}^{SE} \sin(\varphi)]\sigma_{gl}$$

where $\varphi$ is the contact angle of the bridge on both plates. The evolution process is chosen to be a certain combination of iterations (“g” command), refinement (“r” command), and vertex averaging (“V” command). Figures 3.11 and 3.12 demonstrate the force calculated by Surface Evolver and the numerical error of the force versus the
numerical results obtained by Mathematica $\delta F_c = |F_c^{SE} - F_c^{actual}|$. In general the error tends to be higher for lower values of aspect ratio and it may be reduced if more refinement is made.

Figure 3.11: Force calculated by Surface Evolver as a function of $\cos(\theta_c)$ and $a$

Figure 3.12: Relative difference between the force calculated by Surface Evolver and via semi-analytical method as a function of $\cos(\theta_c)$ and $a$
4 Bridge Stability Analysis

4.1 Introduction

Up to this point, our analysis has considered symmetric bridge profiles that satisfy Laplace’s equation and the prescribed contact angle, bridge height, and volume. Missing from this analysis is any discussion of whether the bridge shapes found are stable. That is, whether a perturbation would result in a divergence to another solution, perhaps causing the bridge to spontaneously break.

For bridge stability analysis, either an analytical or a numerical approach may be taken. The later will be considered in the last section of this chapter. For the former, two different stability conditions have been introduced for capillary bridges. Here, we will review the work done on the subject, reproduce and verify some important results from the literature, develop a semi-analytic approach to the problem, and apply the techniques to develop a useful conclusion regarding bridge stability in CFAs.

4.2 Previous Results

The stability of capillary bridges between two parallel plates was first considered mathematically by Vogel [37, 38]. Denote the spacing between the two parallel plates as $h$, the free surface profile as $r(z)$, $z \in [0, h]$, and the rotationally symmetric mean curvature operator as $M$:
\[ M(r) = \frac{1}{2} \left( \frac{r''}{(1 + (r')^2)^{3/2}} - \frac{1}{r(1 + (r')^2)^{3/2}} \right) \]  

(4.1)

where prime indicates a derivative with respect to axial variable \( z \). Then the profile \( r \) must satisfy \( M(r) = \bar{k} \) with the boundary conditions

\[ r'(0) = -\cot \theta_i, \quad r'(h) = \cot \theta_u \]

(4.2)

for some constant \( \bar{k} \), where \( \theta_i \) and \( \theta_u \) are the contact angles with the two plates (denoted \textit{upper} and \textit{lower}). Constant \( \bar{k} \) denotes the mean curvature of the capillary bridge and is related to capillary pressure over interface \( P_c \) as:

\[ \bar{k} = -\frac{P_c}{\sigma_{gl}} \]

(4.3)

Because Eqn. (4.1) and the boundary conditions are axisymmetric, a stable bridge must be rotationally symmetric as well. Vogel proved that for large enough dimensionless volume \( U = \frac{V}{h^3} \), a stable capillary bridge satisfying Laplace equation (4.1) and boundary conditions (4.2) exists. Moreover, this solution approaches a surface of revolution of an arc (satisfying the boundary conditions) uniformly when dimensionless volume tends to infinity. Appendix I contains an analytical approximation of high dimensionless volume bridges as a surface of revolution of circular arc. Vogel’s dimensionless volume is related to the aspect ratio as:

\[ a = \sqrt{\frac{4}{\pi} \sqrt{\frac{1}{U}}} \quad \text{or} \quad U = \frac{4}{\pi} \frac{1}{a^2} \]

(4.4)

Therefore, we may equivalently state that a stable solution for the capillary bridge problem with an arbitrary pair of contact angles exists for small enough aspect ratios and
that this solution approaches a surface of revolution of an arc when the aspect ratio tends to zero.

Furthermore, Vogel has proven that a bridge is stable if the following two conditions are met:

1) The following Sturm-Liouville problem has only one negative eigenvalue:

\[
L(r) = -\left( \frac{r \Psi'}{\left(1 + (r')^2\right)^{3/2}} \right)' - \frac{\Psi}{r\left(1 + (r')^2\right)^{3/2}} = \lambda \Psi, \quad \Psi'(0) = \Psi'(h) = 0 \tag{4.5}
\]

2) For any family of solutions \( r(z; \varepsilon) \) to (4.1) and (4.2) smoothly parameterized by variable \( \varepsilon \) with volume \( V(\varepsilon) \), constant curvature \( \bar{k} = \bar{k}(\varepsilon) \), and fixed contact angles \( \theta_l \) and \( \theta_u \) we have

\[
\frac{\partial \bar{k}}{\partial \varepsilon}(\varepsilon_0) \frac{\partial V}{\partial \varepsilon}(\varepsilon_0) > 0 \tag{4.6}
\]

where \( r(z) = r(z,0) \).

Equation (4.6) may also be written in equivalent form \( \frac{dV}{d\bar{k}}(\varepsilon_0) > 0 \) and this is referred to as the minimum volume condition for stability.

Also presented by Vogel are numerical studies for special cases of contact angles. Vogel’s numerical method used an adaptive grid finite difference routine to find a piecewise linear profile shape; approximated the profile as a cubic spline; and calculated volume and \( \frac{dV}{d\bar{k}} \) using the equations:
The cubic spline was also used to determine the coefficients of (4.5). Once the coefficients are known, the eigenvalues may be computed to verify the stability conditions above.

Vogel also has shown that for large enough dimensionless volume (i.e. small enough aspect ratio) stable bridges with no inflection points may be realized and by decreasing the dimensionless volume (increasing aspect ratio) the bridge remains stable until either condition (1) or (2) is violated. The first condition corresponds to the appearance of an inflection point in the profile at the boundary in symmetric bridges. To further explain this statement, we will turn our attention to the concept of inflection point, exploring this topic with the aid of plots from Langbein’s results [39]. Langbein performed a comprehensive numerical analysis of the cases of equal contact angles and nearly equal contact angles. Langbein presented his results in terms of an alternative dimensionless pressure (called Langbein’s dimensionless pressure in this text and denoted by $p_1$) which is defined as:

$$p_1 = \frac{P_c h}{\sigma_g} = 2 ph / r_0$$

(4.8)

An inflection point is a point along the profile at which an extremum of contact angle would occur if the point was the location of the plate boundary (see Figure 4.1). Inflection points may only be realized within the family of unduloid profiles ($0 < p < 1$). Inflection points and their role in stability will be discussed further in the remainder of this chapter.
Figure 4.2 shows dimensionless volume ($U$) versus Langbein’s dimensionless pressure ($p_l$) for three families of solutions. For large values of dimensionless volume stable bridges may be found with no inflection points. These solutions are the farthest left branch on each plot. As the dimensionless volume is decreased, stability will be lost either due to a sign change of an eigenvalue of the Sturm-Liouville problem or to a change in sign of $dV/\bar{k}$ (i.e., zero slope, Figs. 4.2b,c). The former cause is represented by the bifurcation point in the solution where the 0,1, and 2 inflection branches meet (see Fig 4.2a).

For the case of equal contact angles the two above statements about stability may be translated into a more convenient one: For large enough values of dimensionless volume, stable bridges with no inflection point may be found. Decreasing the dimensionless volume, the bridge will lose stability once a local minimum (Fig. 4.2c) or bifurcation point (Fig 4.2a) is observed in function of dimensionless volume in terms of dimensionless pressure.
It is further shown numerically by Langbein that the bifurcation point is realized before the minimum volume condition is achieved for contact angles above the critical value of 31.15° and therefore appearance of the bifurcation point is the dominant cause of instability for bridges with equal contact angles. This situation is reversed for contact angles below the critical value; a minimum volume will be realized prior to the bifurcation point being reached.

The results presented in Vogel and Langbein’s work are all entirely numerical and certain insights (such as the evolution of profile shape while moving on the curves) are not
achieved. In this work, semi-analytical methods will be employed to re-examine these results and further insight into the evolution of profile shape will be provided.

4.3 Realizable Contact Angles for a Given Kralchevsky Nondimensional Pressure

For our investigation of bridge stability it is useful to consider the range of possible contact angle that occurs along the profile in the different families represented in the Plateau sequence. Considering a section of a profile, we refer to the angle facing the center of the profile as the inner angle and denote it by $\theta_{in}$ . The complementary angle facing toward one of the ends of the profile will be called the outer angle and denoted $\theta_{out}$ (see Figure 4.3). The inner and outer angles are limited for each profile in the Plateau sequence by the angle that occurs at the inflection point. The realizable values for these two angles are illustrated in Figure 4.4 as a function of dimensionless pressure $p$ .

Figure 4.3: Illustration of inner and outer angles on an unduloid profile

Figure 4.4: The possible range of inner and outer angles as a function of dimensionless pressure
It may be observed that for each nondimensional pressure corresponding to the family of unduloids \((0 < p < 1)\) there are minimum and maximum contact angles that may be realized. The extrema of the contact angle for each nondimensional pressure occurs at the inflection point, and is denoted \(\theta_i\). All realizable contact angles occur twice along the profile, once on each side of the inflection point.

### 4.4 Equal Contact Angles

Stability of symmetric capillary bridges (equal contact angles) is of special interest to us due to their appearance in the standard configuration of CFA. In this section we will explain the general behavior of families of symmetric capillary bridges with fixed contact angle when the aspect ratio is varied. This may be achieved in a physical setup by either changing the distance between two plates for a bridge with fixed volume, or by the injection/removal of liquid for a bridge with fixed height. For this investigation, however, the employment of varying aspect ratio is merely a tool for the study of stability.

Here we consider a symmetric capillary bridge with acute contact angles with the aid of Fig. 4.5. Points A and J in the figure indicate axial locations with bridge radius \(r_i\) and point E indicates the neck with bridge radius \(r_0\). At points C and G the bridge radius is the geometric mean \(\sqrt{r_0 r_i}\) and will be inflection points for unduloid profiles and end points for nodoid profiles. Points B, D, F, and H represent all points with the same tangent angle.
Starting with a very low aspect ratio the bridge shape with the prescribed contact angle can only be a nodoid, Case I in Figure 4.5c. In this instance, the profile does not contain an inflection point. When the aspect ratio is increased the profile will switch at $p = 0$. 

Figure 4.5: a) Possible range of contact angle vs. dimensionless pressure. b) Aspect ratio vs. dimensionless pressure. c) Demonstration of profile shape and properties for different values of dimensionless pressure.
from a nodoid to an unduloid (Case II). For any unduloid, a minimum possible value of contact angle exists, this depends on the nondimensional pressure. Furthermore, profiles with zero, one, or two inflection points may be realized for any value of dimensionless pressure \(0 < p < 1\). Since points B, D, F, and H in Figure 4.5c represent points with the same tangent angle, any pair of these points chosen on different sides of the neck E specifies a profile with equal contact angles. It is important to note that only one of these sections, D-F is without any inflection points. Two of the profiles, B-F and D-H, have one inflection point. One profile, section B-H carries two inflection points.

As the aspect ratio is increased, a higher value of the minimum possible contact angle results (see Fig. 4.5a) and this minimum possible angle approaches the actual prescribed contact angle. In terms of profile, increasing aspect ratio causes the points with the same contact angle, B and D, on either side of the inflection point C to approach each other (Case III) and eventually merge with the inflection point (Case IV). In the same manner, points F and H will approach and merge with inflection point H. For the case of equal contact angles, the state of convergence of these points manifests itself as a bifurcation in the solution curve in the parameter space of dimensionless pressure and aspect ratio. This behavior is illustrated in Fig. 4.5b where the 1-inflection and 2-inflection curves branch from the 0 inflection unduloid curve. The same behavior occurs for symmetric bridges with obtuse contact angles when the aspect ratio is increased: the range of realizable contact angles decreases and points with equal contact angle on either side of the inflection point merge together.
The bifurcation point corresponds to a bridge ending at two inflection points. Based on inflection point stability criterion for symmetric bridges, only profiles with no inflection points may be stable. Fig. 4.5b shows that for the contact angle chosen, the stable bridges (no inflection points) can only exist for aspect ratio below the value of this bifurcation point. Recall that having no inflection point is only a necessary condition for the stability of a symmetric bridge. To fully determine stability conditions we must also consider the minimum volume condition. To do so we will begin with Langbein’s results in terms of the parameter space of dimensionless volume and Langbein dimensionless pressure. We then will discuss the transformation from these coordinates into those of aspect ratio and dimensionless pressure.

Figure 4.6 shows dimensionless volume as a function of Langbein’s pressure for equal contact angles varying from 5° to 30°. Only the zero inflection point branch is shown. From Vogel, we know that bridge configurations on the far left (large nondimensional volume) are stable and that stability is maintained for all bridges on each curve to the left of the minimum volume point for that curve. Furthermore, Vogel has shown that a minimum volume point occurs on the zero inflection branch only for contact angles below 31.15°.
In Figure 4.7 the dimensionless volume curves are shown for contact angles ranging from 35° to 150°. It can be observed that a local minimum does not appear on any of the zero inflection branches.

Figure 4.6: Dimensionless volume as a function of Langbein’s pressure for various contact angles. Only the zero inflection branch is shown.

Figure 4.7: Dimensionless volume as a function of Langbein’s pressure for various contact angles. Only the zero inflection branch is shown.
For capillary force actuators, we expect that electrowetting will not yield a contact angle below 35°. Therefore, the minimum volume condition is not of concern and stability is limited by bifurcation, or equivalently the presence of inflection points. The smallest nondimensional volume that may be achieved for a given contact angle is that associated with the bifurcation point. This may be numerically determined for various contact angles to find the minimum nondimensional volume as a function at Langbein’s pressure. This is shown in Figure 4.8 for contact angles from 15° to 175°. Also shown is the limit corresponding to the minimum volume conditions. For contact angles above 31.15°, the bridge is stable if the point corresponding to its nondimensional volume and Langbein pressure lies above the curve shown in Figure 4.8a. For contact angles below 31.15°, stability requires the point to be above the curve in Figure 4.8b.

![Figure 4.8: (a) Inflection point and (b) minimum volume stability limit curves for symmetric capillary bridges](image)

The stability conditions considered here may be translated to conditions in terms of aspect ratio and nondimensional pressure. Figures 4.9 – 4.11 show plots of dimensionless volume vs. Langbein’s pressure along with plots of aspect ratio vs. dimensionless...
pressure for acute, right, and obtuse contact angles. The zero, one, and two inflection point branches are shown. It may be observed that the bifurcation point continues to exist when transformed into the new coordinate system. It is easily shown that the minimum point of the $U - p_i$ curve corresponds to the maximum point of the $a - p$ curve. Thus, the minimum nondimensional volume in Langbein’s coordinate system will translate into the maximum aspect ratio in the new coordinate system. Also because minimum (for obtuse contact angles) or maximum (for acute contact angles) value of the dimensionless pressure occurs at the bifurcation point, we observe infinity slope of aspect ratio vs. dimensionless pressure (zero slope of dimensionless pressure vs. aspect ratio) at the bifurcation point in the new coordinate system.

Figure 4.12 shows aspect ratio as a function of dimensionless pressure for symmetric bridges with contact angle ranging from $35^\circ$ to $175^\circ$. It may be observed that the behaviors of the curves shown in Figures 4.9 - 4.11 are representative of the behavior in general. It is also important to note that the branches for bridges with inflection points exist only for unduloid profiles ($0 < p < 1$) and they disappear once the dimensionless pressure leaves this interval.
Figure 4.9: a) Nondimensional volume vs. Langbein’s nondimensional pressure for symmetric profiles with acute contact angle (60 degrees). b) Corresponding aspect ratio vs. nondimensional pressure. c) Samples of possible profiles of a symmetric bridge with acute contact angles.
Figure 4.10: a) Nondimensional volume vs. Langbein’s nondimensional pressure for symmetric profiles with 90° contact angle. b) Corresponding aspect ratio vs. nondimensional pressure. c) Samples of possible profiles of a symmetric bridge with 90° contact angles.
Figure 4.11: a) Nondimensional volume vs. Langbein’s nondimensional pressure for symmetric profiles with obtuse contact angle (120 degrees). b) Corresponding aspect ratio vs. nondimensional pressure. c) Samples of possible profiles of a symmetric bridge with obtuse contact angles.
Considering the relationship between the two coordinate systems employed, the stability limits developed may alternatively be presented in terms of aspect ratio and dimensionless pressure as shown in Figure 4.13. Here, points below the aspect ratio curve correspond to stable bridges.

Figure 4.12: Aspect ratio- dimensionless pressure curves for contact angles from 35° to 175°

Figure 4.13: Stability limits due to (a) bifurcation (inflection point) and (b) minimum volume conditions. For contact angles above 31.15°, bridges corresponding to points below the curve in (a) are stable. For contact angles below 31.15°, stability is determined by the curve in (b).
For contact angles greater than $35^\circ$ the smallest aspect ratio at which bifurcation can occur is about 1.25. This is more than 12 times larger than aspect ratios that have been typically considered for CFA. Thus, in the standard configuration instability of the bridge is very unlikely. It could result either from a large increase (five times) in the distance between the plates or from the loss of a large fraction (on the order of two-thirds) of the volume.

4.5 Unequal Contact Angles

Bridges with unequal contact angles are particularly important when considering alternative configurations of capillary force actuators. For example, when only one of the two plates is active (i.e., a dielectric covered electrode) the bridge will have unequal contact angles in general. Here we will consider the general behavior of such bridges for different values of contact angles and dimensionless pressure. The stability of such bridges will be examined and measures of stability for such bridges will be introduced. Bridges with unequal contact angle may be classified into three different categories: bridges with acute contact angles at both ends, bridges with obtuse contact angles at both ends, and bridges with acute contact angle at one end and obtuse contact angle at the other. Further, we call a bridge convex if the sum of the contact angles is greater than $180^\circ$ and call it concave if the sum is less than $180^\circ$. Based on this definition, bridges with two acute contact angles would be concave and bridges with two obtuse contact angles would be convex. In the case of bridge with one acute and one obtuse contact angle...
angle both convex \( (\theta_i + \theta_u > 180^\circ) \) and concave \( (\theta_i + \theta_u < 180^\circ) \) bridges may be realized.

### 4.5.1 Acute-Acute

Here we consider a capillary bridge with different acute contact angles at two ends (see Fig. 4.14 Case II). As before, points A and J represent points with bridge radius \( r_i \) and point E represents the neck with bridge radius \( r_0 \). At points C and G the bridge radius is \( \sqrt{r_0 r_i} \) and these points represent the inflection points for unduloid profiles and end points for nodoid profiles. Points B and D indicate all axial locations with tangent angle \( \theta_u \) and points F and H indicate those with tangent angles \( \theta_i > \theta_u \).

For sufficiently low aspect ratios the bridge profile will be a section of a nodoid (Case I, \( p < 0 \)). For any negative value of dimensionless pressure only one section of the nodoid with contact angles \( \theta_i \) and \( \theta_u \) exists. As aspect ratio in increased, the profile will switch from nodoids to unduloids at \( p = 0 \). For dimensionless pressures larger than zero, there will exist four different sections of the unduloid profile that will have the given contact angles \( \theta_i \) and \( \theta_u \). Only one of these profiles will be without any inflection points D-F. Two of the profiles B-F and D-H will have one inflection point and one profile section B-H will include two inflection points. As aspect ratio is further increased, points B and D will approach the inflection point C and points F and H approach the inflection point G (Case II and III, respectively). Because points B and D have a tangent angle smaller than \( F \) and \( H \) (farther from 90 degrees), they will merge with inflection point C before F and H.
(Case III). In the case of two acute contact angles, the smaller contact angle specifies the value of dimensionless pressure beyond which bridges with the given contact angles may no longer be realized (see Fig. 4.14a).

Figure 4.14: Acute-Acute case. a) Possible range of contact angles vs. dimensionless pressure. b) Aspect ratio vs. dimensionless pressure. c) Demonstration of profile shape and properties for different values of dimensionless pressure.
Increasing the aspect ratio beyond that associated with maximum dimensionless pressure results in decreasing dimensionless pressure and a bridge with an inflection point. This bridge will remain stable with increasing aspect ratio until the minimum volume (maximum aspect ratio) condition is met upon the 1 inflection branch of the solution (see Fig. 4.14b).

### 4.5.2 Obtuse-Obtuse

The case of two obtuse contact angles is very similar to the acute-acute case with the principal difference being that dimensionless pressure decreases with increasing aspect ratio on the 0 inflection branch. As before, a nodoid profile exists for very low aspect ratios (Fig. 4.15 Case I, $p > 1$). As aspect ratio is increased the profile will switch from a nodoid to an unduloid at $p = 1$. For unduloid profiles ($p < 1$) there will exist profiles with zero, one and two inflection points. As aspect ratio is further increased, the points with the prescribed contact angle will approach the inflection point. There will be a value of dimensionless pressure at which the larger contact angle (i.e., farther from 90°) will be equal to the tangent angle at the inflection point (Case III). This value is where the 0-inflection branch and the 1-inflection branch will meet (Fig. 4.15b). Similar to the case of acute contact angles, the prescribed contact angle farther from 90 degrees will specify the value of dimensionless pressure below which bridges with the given contact angles may no longer be realized (see Fig. 4.15b).
Figure 4.15: Obtuse-Obtuse case. a) Possible range of contact angles vs. dimensionless pressure. b) Aspect ratio vs. dimensionless pressure. c) Demonstration of profile shape and properties for different values of dimensionless pressure.
4.5.3 Acute-Obtuse

The case of capillary bridges with an acute contact angle on one surface and an obtuse angle on the other is significantly different than the two cases previously considered. Examining the profile shapes shown in Figures 4.14 and 4.15 one may observe that for the acute-acute and obtuse-obtuse cases a section of the profile with points on either side of the center point E is realized as the bridge. But, for the acute-obtuse case the pair of points defining the bridge profile will both fall on one side of the center (Figure 4.16). Another significant difference is the range of dimensionless pressure achievable. In the acute-acute and obtuse-obtuse cases the dimensionless pressure took values either entirely greater than (obtuse-obtuse) or less than (acute-acute) 0.5, the cylinder case. In the acute-obtuse case because of symmetry among unduloid profiles around the cylinder case, each unduloid profile \(0 < p < 1\) may be realized as a section from either an unduloid with neck \(p < 0.5\) or an unduloid with haunch \(p > 1\). However the profiles realized with dimensionless pressures \(p\) and \(1 - p\) would be the same (see Appendix I). It is important to note that although both convex and concave bridges may be realized within family of unduloids with neck or haunch, only convex capillary sections \((\theta_l + \theta_u > 180)\) may be realized within nodoids with haunch and only concave capillary sections \((\theta_l + \theta_u < 180)\) may be realized within nodoids with neck. One rule that will hold independent of whether the contact angles are acute or obtuse is that the limiting value of the dimensionless pressure is determined by the contact angle farther from 90°.
Figure 4.16: Acute-Obtuse case. a) Demonstration of profile shape and properties for different values of dimensionless pressure. b) Possible range of contact angles vs. dimensionless pressure. c) Aspect ratio vs. dimensionless pressure.
In Fig. 4.16a points A and J indicate axial locations with bridge radius $r_1$ and point F marks those with radius $r_0$. Points D and H have bridge radius $\sqrt{r_0 r_1}$ and correspond to inflection points for unduloid profiles. Points B and F mark locations with obtuse outer angle $\theta_i$ while E and C are points with acute inner angle $\theta_a$. Sections C-F and E-F specify profiles with contact angles $\theta_i$ and $\theta_a$. Section E-F is a profile without any
inflection points while C-F contains one. Here, the contact angles are chosen so that the resulting capillary bridge is concave ($\theta_l + \theta_u < \pi$).

With sufficiently low aspect ratio the profile is a nodoid with neck (Case $i$, $p < 0$) and only one section with specified contact angles, E-F, may be realized. It is important to note that only concave profiles may be realized as a section of a nodoid with neck. As aspect ratio is increased the profile shape will change to an unduloid after the dimensionless pressure reaches zero. Two profiles, one with no inflection and one with a single inflection point may be realized at these pressures. As the aspect ratio in increased points B, C, E, and F all approach the inflection point D. In the figure contact angle $\theta_u$ is farther from $90^\circ$ than $\theta_l$ i.e., $|\pi - \theta_l| < |\pi - \theta_u|$. As a result points C and E will merge to the inflection point first. The limiting value of dimensionless pressure ($p_{ivp}$) will depend on the value of contact angle that is farthest from $90^\circ$ (in the figure, $\theta_u$).

As observed above, concave bridges with both acute and obtuse angles may be realized within the capillary bridge profiles with neck. Any section realizable within the family of unduloids with neck may also be realized within the family of unduloids with haunch. Fig. 4.16f shows the sections of unduloids with haunch that have the same contact angles, $\theta_l$ and $\theta_u$, as before. Points A, D, G, H, and I are defined as before. However in this case points B and F represent points with acute outer angle $\theta_u$ and points C and E represent points with obtuse inner angle $\theta_l$. Thus sections C-B and E-B specify profiles with contact angles $\theta_l$ and $\theta_u$ with no inflection and one inflection, respectively. In this
case, when increasing the aspect ratio the dimensionless pressure will decrease until the limit value ($p_{iv}$). This value of pressure is (i.e., $p_{iv} = 1 - p_{iv}$) the mirror image of the former limit with respect to $p = 0.5$.

Plots similar to those for concave capillary bridges may be constructed for convex bridges with unequal contact angles. The aspect ratio-dimensionless pressure curve for convex bridges would be very similar to the one of concave bridges. The only difference would be that the right half of the curve in the unduloid region will extend (toward the nodoids with haunch, $p > 1$) rather than the left half extending (toward nodoids with neck, $p < 0$).

When both contact angles are greater than $31.15^\circ$, the stable 0-inflection branch will connect to a stable 1-inflection branch at the extreme value of dimensionless pressure, $p$. Thus, stable capillary bridges with either zero or one inflection point may be found. Figures 4.17 - 4.20 demonstrate the evolution of aspect ratio vs. dimensionless pressure and nondimensional volume vs. Langbein’s dimensionless pressure for selected pairs of contact angles chosen.
Figure 4.17: Axisymmetric bridge solutions in two coordinate systems for profiles with two acute contact angles

\[ \theta_i = \theta_u = 60 \quad \theta_i = 60, \theta_u = 75 \quad \theta_i = 60, \theta_u = 90 \]

Figure 4.18: Axisymmetric bridge solutions in two coordinate systems for profiles with two obtuse contact angles

\[ \theta_i = \theta_u = 120 \quad \theta_i = 120, \theta_u = 105 \quad \theta_i = 120, \theta_u = 90 \]
Figure 4.19: Axisymmetric bridge solutions in two coordinate systems for concave profiles with one acute and one obtuse contact angles

\[ \theta_i = 60, \theta_u = 100 \]
\[ \theta_i = 60, \theta_u = 110 \]
\[ \theta_i = 60, \theta_u = 120 \]

Figure 4.20: Axisymmetric bridge solutions in two coordinate systems for convex profiles with one acute and one obtuse contact angles

\[ \theta_i = 120, \theta_u = 80 \]
\[ \theta_i = 120, \theta_u = 70 \]
\[ \theta_i = 120, \theta_u = 60 \]
Capillary bridges with unequal contact angles will be stable for sufficiently low (high) aspect ratios (dimensionless volume). Stability will be maintained along the solution curve as aspect ratio is increased until the minimum volume (maximum aspect ratio) condition is reached (bifurcation will not occur for unequal contact angles).

The stability condition for bridges may be expressed either in terms of aspect ratio or nondimensional volume. For aspect ratio, the stability condition is that the bridge aspect ratio is less than a critical value for the given contact angles, i.e., \( a < a^*(\theta_1, \theta_u) \). This critical value corresponds to the maximum aspect ratio achieved on the family of solutions starting with the 0-inflection branch. For nondimensional volume, the stability condition is that the nondimensional volume must be greater than a critical value for the given contact angle, i.e. \( U > U^*(\theta_1, \theta_u) \). The critical value, of course, is the minimum value of \( U \) achieved on the family of solutions emanating from the 0-inflection branch.

The critical values of nondimensional volume and aspect ratio were calculated for bridges with unequal contact angles along with the bifurcation stability limits calculated for bridges with equal contact angles. These are plotted together for contact angles, \( \theta_1 \) and \( \theta_u \), between 30° and 150° in Figures 4.21- 4.23. It may be observed that the minimum critical nondimensional volume (as well as maximum stability limit in aspect ratio plot) over all pairs of contact angles occurs at \( \theta_1 = \theta_u = 90^\circ \). That is, this pair of contact angles permits the stable bridge with the smallest nondimensional volume (or largest aspect ratio). Also maximum critical nondimensional volume over the range of angles examined
occurs for the case of $\theta_l = 30^\circ$ and $\theta_u = 150^\circ$. At this pair of contact angles the bridge is least stable (over the range of angles examined). That is, this bridge has the smallest aspect ratio where instability may result. However, the minimum critical aspect ratio found ($a^* = 0.283$) is three times larger than the typical aspect ratios considered for capillary force actuators. Thus we expect in most applications bridge stability will not be a concern.

![Figure 4.21: Nondimensional volume stability limit for capillary bridges with contact angles $\theta_l$ and $\theta_u$](image1)

![Figure 4.22: Aspect ratio stability limit for capillary bridges with contact angles $\theta_l$ and $\theta_u$](image2)
4.6 Numerical Methods for Stability

Another approach to examine bridge stability employs Surface Evolver. We briefly introduce the reader to these ideas here, although they are employed only for checking results in this dissertation. By checking the sign and magnitude of the Hessian matrix eigenvalues stability can be determined. Consider a particular configuration of a surface numerically expressed and assume a numerical perturbation is applied. The Hessian is the square matrix of second derivatives of energy to perturbations. Each eigenvalue expresses the response of the surface along the eigenvector corresponding to that eigenvalue. Positive eigenvalues indicate a stable equilibrium and negative eigenvalues indicate instability along the direction of the eigenvector. In addition the higher the magnitudes of the positive eigenvalues are, the greater the restoring force will be, which represents
faster recovery from this particular deviation from equilibrium shape. Figure 4.24 shows the deformations of bridge along few of the eigenvectors.

Figure 4.24: Capillary bridge perturbation along three eigenvectors of Hessian matrix
5 Alternative Configurations of CFA

5.1 Introduction

Depending on the application, alternative CFA configurations to the standard configuration examined in Chapter 2 may prove to be beneficial. For example, electrodes on both sides of the capillary bridge may be prohibited in certain applications due to manufacturing or materials restrictions. In some cases, it may be desirable to pin the contact line on the passive (i.e., no electrode) side. Gradients of surface energy may be desirable on either the passive or active sides. Furthermore, certain configurations hold the promise of increasing force production. In this chapter, these alternative configurations are examined.

5.2 Parallel Plate CFA with Alternate Boundary Conditions

In order to investigate more complex device configurations, the nondimensional equations introduced for the standard configuration case will be our starting point. Equations employed for standard configuration are only capable of characterizing bridges with equal contact angles. In this chapter, bridges with unequal contact angles may be observed due to the different boundary conditions on the two plates. Therefore, for an analytical characterization of such bridges more general expressions for bridge height and volume will be introduced. Since the profiles in Delaunay sequence are the only solutions to the axisymmetric Laplace equation, the profile of any capillary bridge independent of
the boundary conditions imposed must be realized as a scaled version of a section of one of the shapes within the sequence. Here we consider all of the scenarios that may occur when such a section may be chosen from any of the shapes within the Delaunay sequence. For a bridge with capillary pressure $p$ and nondimensional contact radii $\rho_i$ and $\rho_u$, we introduce new nondimensional functions $\zeta_j(\rho_i, \rho_u, p)$ and $\nu_j(\rho_i, \rho_u, p)$ for bridge height and volume respectively, and express these functions in terms of the nondimensional functions employed in the standard configuration. (Here subscript $j$ represents the scenario number.). Irrespective of bridge shape, three different scenarios may be considered: (1) The bridge section contains the neck (haunch), (2) The bridge section does not contain the neck (haunch) and $\rho_u > \rho_i > \rho_0$, and (3) The bridge section does not contain the neck (haunch) and $\rho_i > \rho_u > \rho_0$. The graphical illustration of the three scenarios and analytical expressions for two functions introduced in each case are listed in Table 5.1.

Depending on the boundary conditions imposed by the configuration, the analysis of problem may be reduced to either one or two dimensional root finding. The different boundary condition problems that are of interest are listed in Table 5.2. Total of six cases with axisymmetric boundary conditions are considered. Case 1 in the table is the standard configuration previously investigated in Chapter 2.
Table 5.1: Nondimensional height and volume for nonsymmetric capillary bridges

<table>
<thead>
<tr>
<th>Condition</th>
<th>Equation 1</th>
<th>Equation 2</th>
<th>Equation 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho_0 &lt; \rho_u, \rho_i &lt; \rho_1$</td>
<td>$\zeta_1 = \zeta(\rho_1, p) + \zeta(\rho_u, p)$</td>
<td>$\zeta_2 = \zeta(\rho_u, p) - \zeta(\rho_1, p)$</td>
<td>$\zeta_3 = \zeta(\rho_1, p) - \zeta(\rho_u, p)$</td>
</tr>
<tr>
<td>$\rho_1 &gt; \rho_u &gt; \rho_i &gt; \rho_0$</td>
<td>$\nu_1 = \frac{1}{2}[\nu(\rho_1, p) + \nu(\rho_u, p)]$</td>
<td>$\nu_2 = \frac{1}{2}[\nu(\rho_u, p) - \nu(\rho_1, p)]$</td>
<td>$\nu_3 = \frac{1}{2}[\nu(\rho_1, p) - \nu(\rho_u, p)]$</td>
</tr>
<tr>
<td>$\rho_0 &lt; \rho_u &lt; \rho_i &lt; \rho_1$</td>
<td>$\rho_0 &gt; \rho_u, \rho_i &gt; \rho_1$</td>
<td>$\rho_0 &gt; \rho_i &gt; \rho_u &gt; \rho_1$</td>
<td>$\rho_0 &gt; \rho_u &gt; \rho_i &gt; \rho_1$</td>
</tr>
</tbody>
</table>
Table 5.2: Boundary condition problems for alternative configurations of CFA

<table>
<thead>
<tr>
<th>Case</th>
<th>Given Parameters</th>
<th>Unknows</th>
<th>Equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$h$: bridge height $\theta_0$: contact angle $V$: bridge volume</td>
<td>$p$: capillary pressure $\rho_c$: dimensionless radius $\kappa$: scale factor</td>
<td>$z(\rho_c, p) = \kappa \zeta(\rho_c, p) = \frac{h}{2}$ $\rho_c = P(\theta_0, p)$ $V = \kappa^3 v(\rho_c, p)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\text{Solve}\left[ \frac{h^3 v(P(\theta_0, p), p)}{[2 \zeta(P(\theta_0, p), p)]^3} - V = 0 \right] \text{ for } p$</td>
</tr>
<tr>
<td>2</td>
<td>$h$: bridge height $\theta_l$: lower plate contact angle $\theta_u$: upper plate contact angle $V$: bridge volume</td>
<td>$p$: capillary pressure $\rho_l$: dimensionless radius at lower plate $\rho_u$: dimensionless radius at upper plate $\kappa$: scale factor</td>
<td>$\rho_l = P(\theta_l, p)$ $\rho_u = P(\theta_u, p)$ $\kappa \zeta_j(\rho_l, \rho_u, p) = h$ $V = \kappa^3 v_j(\rho_l, \rho_u, p)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\text{Solve}\left[ \frac{h^3 v_j(P(\theta_l, p), P(\theta_u, p), p)}{[\zeta_j(P(\theta_l, p), P(\theta_u, p), p)]^3} - V = 0 \right] \text{ for } p$</td>
</tr>
<tr>
<td>3</td>
<td>$h$: bridge height $\theta_0(r_c)$: contact angle as a function of radius $V$: bridge volume</td>
<td>$p$: capillary pressure $\rho_c$: dimensionless contact radius $r_c$: contact radius $\kappa$: scale factor</td>
<td>$z(\rho_c, p) = \kappa \zeta(\rho_c, p) = \frac{h}{2}$ $\rho_c = P(\theta_0(r_c), p)$ $V = \kappa^3 v(\rho_c, p)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\text{Solve}\left[ { \frac{h^3 v(P(\theta_0(r_c), p), p)}{[2 \zeta(P(\theta_0(r_c), p), p)]^3} - V = 0, \right.$ $\frac{2 \zeta(P(\theta_0(r_c), p), p)}{h} - \frac{P(\theta_0(r_c), p)}{r_c} = 0 } \text{ for } { p, r_c }$</td>
</tr>
</tbody>
</table>
Table 5.2 Cont.: Boundary condition problems for alternative configurations of CFA

<table>
<thead>
<tr>
<th>Case</th>
<th>Given Parameters</th>
<th>Unknowns</th>
<th>Equations</th>
</tr>
</thead>
</table>
| 4    | $h$: bridge height  
$r_c$: bridge radius  
$V$: bridge volume | $p$: capillary pressure  
$r_c$: dimensionless radius  
$\kappa$: scale factor | $\kappa \rho_c = r_c$,  
$\kappa \zeta (\rho_c, p) = h/2$  
$V = \kappa^3 u(\rho_c, p)$  
$\text{Solve}[\{u(\rho_c, p) - V \frac{r_c^3}{\rho_c^3} = 0, \frac{r_c}{\rho_c} \zeta (\rho_c, p) - h/2 = 0\}]$ for $\{p, \rho_c\}$ |
| 5    | $h$: bridge height  
$r_i$: bridge radius at lower plate  
$r_u$: bridge radius at upper plate  
$V$: bridge volume | $p$: capillary pressure  
$r_i$: dimensionless radius on one end  
$r_u$: dimensionless radius at upper plate  
$\kappa$: scale factor | $\kappa \rho_i = r_i$,  
$\kappa \rho_u = r_u$,  
$\kappa \zeta (\rho_i, \rho_u, p) = h$  
$V = \kappa^3 u_j(\rho_i, \rho_u, p)$  
$\text{Solve}[\{u_j(\frac{r_i}{\kappa}, \frac{r_u}{\kappa}, p) - V \frac{r_i^3}{\rho_i^3} = 0, \frac{r_i}{\rho_i} \zeta (\rho_i, \rho_u, p) - \frac{h}{\kappa} = 0\}]$ for $\{p, \kappa\}$ |
| 6    | $h$: bridge height  
$r_i$: bridge radius at lower plate  
$\theta_u$: contact angle at upper plate  
$V$: bridge volume | $p$: capillary pressure  
$r_i$: dimensionless radius at lower plate  
$r_u$: dimensionless radius at upper plate  
$\kappa$: scale factor | $\kappa \rho_i = r_i$,  
$\kappa \zeta (\rho_i, \rho_u, p) = h$  
$V = \kappa^3 u_j(\rho_i, \rho_u, p)$  
$\text{Solve}[\left\{ \frac{r_i}{\rho_i} \right\}^3 u_j(\rho_i, P(\theta_u, p), p) - V = 0,  
\left( \frac{r_i}{\rho_i} \right) \zeta (\rho_i, P(\theta_u, p), p) - h = 0\}]$ for $\{p, \rho_i\}$ |
5.3 Case 2: Parallel Plate Configuration with Unequal Contact Angles

5.3.1 Introduction

The first alternative configuration to be considered is the parallel plate case with different contact angles on the two plates. The contact angle on one plate is considered fixed (a passive surface) while the other contact angle is variable (an active surface). Using the analytical expressions introduced in Section 2.3 a one-dimensional root-finding problem may be formed and solved to determine the bridge shape and the capillary force produced. Here we will consider different values for the fixed contact angle and compare the efficiency of the device to the standard configuration case.

5.3.2 Approach

Given nondimensional pressure and contact angles on the plates, $\theta_i$ and $\theta_u$, nondimensional contact radii, $\rho_i$ and $\rho_u$, may be calculated as:

$$\rho_i = P(\theta_i, p)$$

$$\rho_u = P(\theta_u, p)$$

where $P(\varphi, p)$ gives the nondimensional radius for given pair of nondimensional pressure and contact angle and is defined in Section 2.3. Scale factor $\kappa$ may be found by equating the profile height $h$ with the scale factor times the nondimensional height of the bridge $\zeta_j(\rho_i, \rho_u, p)$:

$$\kappa \zeta_j(\rho_i, \rho_u, p) = h$$
Solving for $\kappa$ yields:

$$\kappa = \frac{h}{\zeta_j}(\rho_l, \rho_u, p) \quad (5.4)$$

The volume of the bridge may be calculated as:

$$V = \kappa^3 \nu_j(\rho_l, \rho_u, p) \quad (5.5)$$

Substitution for $\kappa$ from Eqn. (5.4) and for $\rho_l$ and $\rho_u$ from Eqns (5.1) and (5.2) yields the equation for the bridge volume in terms of dimensionless pressure, contact angles and bridge height:

$$V = \frac{h^3 \nu_j(P(\theta_l, p), P(\theta_u, p), p)}{[\zeta_j(P(\theta_l, p), P(\theta_u, p), p)]^3} \quad (5.6)$$

Dimensionless pressure is the only unknown in Eqn. (5.6) and therefore it may be calculated by one-dimensional root-finding for the given bridge height, volume, and contact angles. Having the value of dimensionless pressure, the parameters governing the bridge shape, $\rho_l$, $\rho_u$, $\rho_0$, $\rho_1$, and $\kappa$ may be determined. Capillary force then may be calculated using Eqn. (2.30).

### 5.3.3 Results and Discussion

An analytical approximation of the force in a pancake shape bridge with unequal contact angles may be derived by considering a bridge as a surface of revolution of a circular arc. The total potential energy of such a bridge is given by:

$$E = \sigma_{glj}(S_{glj} - (\cos \theta_l + \cos \theta_u)S_{ls}) + 2\sigma_{sg}S_s \quad (5.7)$$

Therefore, the equation of force may be derived by taking a derivative of this energy with respect to a virtual displacement of the plates:
Considering the bridge with low aspect ratio as a cylinder with radius $r$ and height $h$ we may obtain:

$$\frac{\partial S_t}{\partial h}\bigg|_{V\text{ fixed}} = \pi r \tag{5.9}$$

$$\frac{\partial S}{\partial h}\bigg|_{V\text{ fixed}} = -\pi r \left(\frac{r}{h}\right) \tag{5.10}$$

Substitution of these into Eqn. (5.8) yields:

$$F_c = -\frac{\partial}{\partial h} E = -\sigma_{gl} \pi r \left[1 + (\cos \theta_i + \cos \theta_u) \left(\frac{r}{h}\right)\right] \tag{5.11}$$

One may consider the change in force with respect to change in cosine of varying contact angle $\theta_u$ to obtain:

$$\frac{dF_c}{d \cos \theta_u} = -\frac{\partial}{\partial h} E = -\sigma_{gl} \pi r \left(\frac{r}{h}\right) \tag{5.12}$$

This shows the independence of this ratio from the fixed contact angle $\theta_i$. Also, a comparison of this result with that of the standard configuration indicates that the amount of change in force is expected to be only half that with two active surfaces. To numerically validate this behavior, we let contact angle $\theta_i$ take on three different values $(60^\circ, 90^\circ, 120^\circ)$ and the contact angle on the active plate, $\theta_u$, take discrete values in the intervals $[30^\circ, 115^\circ]$, $[60^\circ, 150^\circ]$ and $[65^\circ, 150^\circ]$ respectively. The height and volume are the same in all the cases and equal to $100 \mu m$ and $7.85 \times 10^7 \mu m^3$ respectively. These values are chosen to be similar to those used in the parallel plate case with equal contact angles examined in Chapter 2. The force as well as the dimensionless pressure is plotted
in Figure 5.1 as a function of the cosine of the varying contact angle $\theta_v$ for three values of fixed contact angle $\theta_f$.

A comparison of the amount of force in these cases with those in the case of equal contact angles confirms that the change in force with respect to change in contact angle cosine is in fact half of the amount for the standard configuration. However it is important to note that with special arrangement of device configuration the voltage required to make the same change in contact angle cosine may be reduced to half the amount required in the standard configuration. Table 5.3 summarizes the maximum force and change in force for the cases examined.
Table 5.3: Comparison of force for bridges with equal and unequal contact angles

| \( \theta_i \) | \( \theta_u \) | \( \Delta \cos(\theta_u) \) | Force Range (\( \mu N \)) | \( \Delta F \) (\( \mu N \)) | \( |F|_{\text{max}} \) | \( \Delta F / \Delta \cos(\theta_u) \) (\( \mu N \)) |
|---|---|---|---|---|---|---|
| 60° | [60°,115°] | 1.28864 | [-870, -140] | 730 | 870 | 490 |
| 90° | [60°,150°] | 1.73205 | [-590, 390] | 980 | 590 | 566 |
| 120° | [65°,150°] | 1.28864 | [-60, 670] | 730 | 670 | 490 |
| \( \theta_1 = \theta_2 \in [45°, 120°] \) | 1.20711 | [-900, 460] | 1360 | 900 | 1040 |

Design charts for the passive contact angles of 70°, 90°, and 110° are shown in Figures 5.2-5.4, respectively.

Figure 5.2: Design chart for passive contact angle of 70°
Figure 5.3: Design chart for passive contact angle of $90^\circ$

Figure 5.4: Design chart for passive contact angle of $110^\circ$
5.4 Case 3: Parallel Plate Configuration with Contact Angle Gradient

5.4.1 Introduction

One interesting configuration employs a gradient of surface energy (i.e. hydrophilicity/hydrophobicity) upon the parallel plates. Recall that Young’s equation dictates that the interfacial energy determines the contact angle of a liquid on a solid surface as:

\[
\sigma_{ls} + \sigma_{gl} \cos(\theta_0) = \sigma_{gs}
\]  

(5.13)

where \(\sigma_{ls}\), \(\sigma_{gl}\), and \(\sigma_{gs}\) denote the interfacial free energy, gas/liquid surface energy and solid/gas surface energy respectively.

Chaudhury and Whitesides [40] examined a sessile drop upon a surface with a gradient in contact angle with two sides of the contact line experiencing different surface energies due to a gradient in surface energy. A net force in the direction of the lower contact angle was produced resulting in drop motion. Considering the problem from an energy point of view, the drop moves to decrease its overall surface energy. It was demonstrated that such a gradient can cause a drop to move up hill. Here we consider the effect of contact angle gradient in the axisymmetric parallel plate case and study the effect of employing it on device performance.
### 5.4.2 Approach

One (direct) approach to solving this problem would be to extract and solve the equations governing the configuration. Denote the contact radius and dimensionless contact radius as \( r_c \) and \( \rho_c \) respectively. These two are related to each other as:

\[
  r_c = \kappa \rho_c \quad (5.14)
\]

where \( \kappa \) is the scale factor. The contact angle gradient on the surface may be denoted by \( \theta_0(r_c) \). Therefore, the dimensionless contact radius may be related back to contact radius as:

\[
  \rho_c = \frac{1}{\kappa} \frac{1}{P(\theta_0(r_c), p)} \quad (5.15)
\]

Substitution of Eqn. (5.14) into (5.15) and solving for \( \kappa \) yields:

\[
  \kappa = \frac{r_c}{P(\theta_0(r_c), p)} \quad (5.16)
\]

Alternatively, \( \kappa \) may be found as the ratio of bridge height and nondimensional height of the bridge:

\[
  \kappa = \frac{h}{2} \zeta (\rho_c, p) \quad (5.17)
\]

Substitution of Eqn. (5.15) into (5.17) yields:

\[
  \kappa = \frac{h}{2} \zeta (P(\theta_0(r_c), p), p) \quad (5.18)
\]

Equating equations (5.16) and (5.18) gives an equation in terms of nondimensional pressure \( p \) and contact radius \( r_c \) only.

\[
  \frac{h}{2} \zeta (P(\theta_0(r_c), p), p) = \frac{r_c}{P(\theta_0(r_c), p)} \quad (5.19)
\]

In addition volume of the bridge may be calculated as:
Substitution for \( \rho_c \) and \( \kappa \) from Eqns. (5.15) and (5.16) respectively, yields a second equation in terms of only two variables: dimensionless pressure \( p \) and contact radius \( r_c \):

\[
V = \kappa^3 \nu(\rho_c, p) \tag{5.20}
\]

Solving (5.19) and (5.21) simultaneously, one may find the dimensionless pressure and contact radius. This would require two-dimensional root-finding. An alternative approach may be employed which takes advantage of the data available for the standard configuration and requires solving a simpler boundary condition problem.

Consider the contact angle and radius of the equilibrium shape of the bridge with a gradient of contact angle to be \( \theta_0 \) and \( r_c \), respectively. The final shape in this case would be the same as the equilibrium shape of a bridge with the same volume and spacing between the plates but with a constant contact angle of \( \theta_0 \) across the plate (i.e. without gradient). This suggests employing the results from case 1 examined in Chapter 2 to solve the gradient case. To do so we consider a bridge with the same volume and height. The problem of uniform contact angle boundary conditions may be solved for a range of contact angles and the contact radius may be recorded to define a numerically evaluated function of contact angle in terms of contact radius, i.e. \( \hat{\theta}(r_c) \). The intersection of this function with the function of the actual energy gradient of the surface \( \theta_0(r_c) \), yields the realized contact angle and radius.
Independent of the approach taken, the effect of applied voltage may be considered as a uniform shift in the cosine of contact angle gradient function (i.e. \( \cos(\theta_0(r_c)) \)) due to Young-Lippmann equation:

\[
\cos(\theta_c) = \cos(\theta_0) + \frac{\varepsilon}{2l_d \sigma_g l}\left(\frac{v}{2}\right)^2
\]  

(5.22)

Mathematically, this relation will hold as long as the right hand side of the equation below is less than 1 (we ignore contact angle saturation here.). So, to make this equation accurate for the gradient functions we write:

\[
\cos(\theta_c(r_c)) = \min\{\cos(\theta_0(r_c)) + \frac{\varepsilon}{2l_d \sigma_g l}\left(\frac{v}{2}\right)^2, 1\}
\]  

(5.23)

Once the bridge shape is known, the force exerted may be found using Eqn. (2.30).

### 5.4.3 Results and Discussion

Here we will investigate the effect of surface energy gradient and its intensity with an example. We use the second (alternative) approach discussed because more insight about the effect of gradient may be gained this way. Because the effect of an applied voltage may be translated as simply adding a constant to contact angle cosine, the discussion and graphs here will be in terms of \( \cos(\theta_0(r_c)) \) and \( \hat{\theta}(r_c) \) instead of \( \theta_0(r_c) \) and \( \hat{\theta}(r_c) \), respectively. Considering the typical values of bridge height and volume used throughout this dissertation (\( h = 100 \mu m \) and \( V = 7.85 \times 10^7 \mu m^3 \)) the function \( \theta_0(r_c) \) may be constructed by solving the uniform contact angle case for a range of contact angles.

Figure 5.5 shows \( \cos(\theta_0(r_c)) \) for the range considered.
The surface energy gradient is chosen for this example so that the cylinder case is always realized at no voltage. This means at zero voltage $\cos(\theta_0(r_c))$ should cross $\cos(\hat{\theta}(r_c))$ at $r_c = 500$. Seven different contact angle gradient slopes, $\beta$, are considered (Fig. 5.3).

For the contact angle gradients shown in Figure 5.6, contact radius and force vs. change in contact angle cosine are plotted in Figure 5.7.

Figure 5.5: Cosine of contact angle vs. contact angle radius

Figure 5.6: Contact angle gradients with different slopes
It may be observed that the greater positive gradient slope (hydrophobic center and hydrophilic surrounding) causes more change in contact radius and capillary force with respect to change in contact angle. On the other hand greater negative gradient slope (hydrophilic center and hydrophobic surrounding) causes less change in contact radius and capillary force with respect to change in contact angle. It is important to note that surface energy gradient remains effective as long as bridge contact radius falls inside the gradient region. For high gradient slope, the transition region between the hydrophobic region (center) and the hydrophilic region (exterior) is narrower. As a result less change in bridge height is permissible if the surface energy gradient effect is not to be lost.

Figure 5.7: Contact angle radius and capillary force vs. change in contact angle cosine for different gradient slopes
5.5 Case 4: Parallel Plate Configuration with Bridge Pinned at Both Ends

5.5.1 Introduction

Pinned boundary condition at the contact line may be achieved by creating either a topological feature or an abrupt change in surface energy, see Figure 5.8. Independent of the method used for pinning, the mathematical description of the boundary condition for a pinned bridge in axisymmetric configuration is a constant bridge radius on the plate surface. While this configuration is not of particular interest for CFA, it has been one of the earliest and most-extensively investigated configurations. It also provides a useful starting point for the investigation of a bridge pinned on one plate with a variable contact angle (via electrowetting) on the other, which will be examined in the next section.

![Figure 5.8: Capillary bridge pinned on both sides](image)

5.5.2 Approach

From our previous analysis, three parameters, contact angle, dimensionless pressure, and scale factor, dictate a unique contact radius and volume for a capillary bridge. These parameters are used to construct the solutions presented in this section for the pinned
case. Denote the pinned radius and dimensionless pinned radius as $r_c$ and $\rho_c$, respectively. These two are related to each other as:

$$r_c = \kappa \rho_c$$

(5.24)

where $\kappa$ denotes the scale factor. The equation above may be solved for $\kappa$ to yield:

$$\kappa = \frac{r_c}{\rho_c}$$

(5.25)

Height and volume of the bridge may be calculated as:

$$h = 2\kappa \zeta(\rho_c, p)$$

(5.26)

$$V = \kappa^3 \nu(\rho_c, p))$$

(5.27)

Substitution from Eqn. (5.25) for $\kappa$ yields two equations in terms of only dimensionless pressure $p$ and dimensionless pinned radius $\rho_c$:

$$h = 2\frac{r_c}{\rho_c} \zeta(\rho_c, p)$$

(5.28)

$$V = \left(\frac{r_c}{\rho_c}\right)^3 \nu(\rho_c, p))$$

(5.29)

A two dimensional root finding routine may be used to determine the simultaneous solution to (5.28) and (5.29). Once the parameters governing the bridge shape are found, one may use Eqn. (2.30) to find the capillary force.

### 5.5.3 Results and Discussion

New dimensionless parameters appropriate for this particular configuration will be introduced to elucidate the results. Given a drop volume $V$, spacing $h$, and pinned diameter $D$ one may define pinned aspect ratio $a$ and volume ratio $U$ as follows:
\[ a^\circ = \frac{h}{D} \]  
\[ U^\circ = \frac{V}{\pi D^3 H / 4} \]

Note that \( U^\circ = 1 \) for a cylindrical bridge. Employing these two parameters as our coordinates, a map of bridge families with important limiting cases may be developed [41], (see Figure 5.9). Whether the bridge has a neck or a haunch may be determined via the magnitude of \( U^\circ \) (i.e. \( U^\circ < 1 \) for bridges with neck, \( U^\circ = 1 \) for cylindrical bridges, and \( U^\circ > 1 \) for bridges with haunch). The cylinder configurations are marked in Fig. 5.9 with the horizontal line \( U^\circ = 1 \). The stable realizable profiles will fall between the blue curve on the left and the red curve. The surrounding curve consists of three constant contact angle contours and a stability limit curve. Two contact angle contours form the left end of the blue region with the part below the \( U^\circ = 1 \) line corresponding to \( D_0^\circ \) and the part above the \( U^\circ = 1 \) line corresponding to \( D_{180^\circ}^\circ \). Contour of 90 degree contact angle falls on the dark line in part and branches upward and downward to form the right upper and lower section of the blue line. The blue curve at the lower end of the stability region represents the minimum volume instability for contact angles below \( 90^\circ \). The red curve separates the profiles with inflection points on the right from the ones without inflection points on the left. Finally, the purple curve corresponds to catenoid ( \( p = 0 \) ) and sphere ( \( p = 1 \) ) profiles. Contours of constant contact angle in this new coordinate system are illustrated in Figure 5.10.
Profiles without Contact Angles

Profiles with Unduloids

Figure 5.9: Stability limits of symmetric capillary bridges in pinned coordinate system

Figure 5.10: Illustration of contact angle contours in pinned coordinate system
To calculate the force between the plates for pinned bridges we define the dimensionless capillary force as:

$$f^o = \frac{F}{\sigma_{gl} h} = -2\pi \frac{r^o}{h} (1 - p)$$  \hspace{1cm} (5.32)

Varying the dimensionless pressure and contact angle, numerically-defined functions for $\rho(a^o, U^o)$, $\theta(a^o, U^o)$ and $f^o(a^o, U^o)$ may be obtained. Figure 5.11 show these three functions for values of $a^o$ ranging from 0.08 to 0.15.

Figure 5.11 indicates that a significant change in contact angle, dimensionless pressure, and consequently force may be achieved, with a small variation in bridge volume. For example for a capillary bridge with a height of 100 $\mu$m and a pinned radius of 500 $\mu$m ($a^o = 0.10$) the change in force obtained by removal or addition of 10% of the volume to a cylinder bridge would be $\pm 1mN$. Thus an interesting alternative approach to capillary force microactuation might be to employ volume perturbation (perhaps via evaporation) to a pinned bridge. Since the contact line is fixed in the pinned case, contact angle hysteresis will be essentially zero for such a device. Figure 5.12 shows the change in profile shape resulting from $\pm 10\%$ variation in volume.
Figure 5.11: Illustration of change in (a) capillary pressure; (b) contact angle (b), and (c) nondimensional force vs. volume ratio for aspect ratios from 0.08 to 0.15

Figure 5.12: Change in bridge shape of a pinned capillary bridge with volume ratio (different scales on axes)
5.6 Case 5: Parallel Plate Configuration with Bridge Pinned at Both Ends with Unequal pinned radii

Like Case 4, Case 5 has both contact lines pinned and hence electrowetting is impossible. Therefore, for this case, only the analytical equations needed to solve the boundary condition problem will be introduced. Given pinned radii at both ends, \( r_i \) and \( r_u \), the dimensionless pinned radii may be found as:

\[
\rho_i = \frac{1}{\kappa} r_i \tag{5.33}
\]

\[
\rho_u = \frac{1}{\kappa} r_u \tag{5.34}
\]

where \( \kappa \) is the scale factor. Height and volume of the bridge may be calculated as:

\[
h = \kappa \zeta_j (\rho_i, \rho_u, p) \tag{5.35}
\]

\[
V = \kappa^3 \nu_j (\rho_i, \rho_u, p) \tag{5.36}
\]

Nondimensional radii, \( \rho_i \) and \( \rho_u \), may be substituted from Eqns. (5.33) and (5.34) to yield:

\[
h = \kappa \zeta_j \left( \frac{1}{k} r_i, \frac{1}{k} r_u, p \right) \tag{5.37}
\]

\[
V = \kappa^3 \nu_j \left( \frac{1}{k} r_i, \frac{1}{k} r_u, p \right) \tag{5.38}
\]

Eqns. (5.37) and (5.38) are only in terms of dimensionless pressure and scale factor and may be solved using a two-dimensional root-finding routine. Parameters governing bridge shape may be found using the solution and the capillary force may be calculated using Eqn. (2.30).
5.7 Case 6: Parallel Plate Configuration with Bridge Pinned on One Plate

5.7.1 Introduction

We now examine the case of a bridge pinned on one plate with electrowetting (moving contact line) on the other, see Fig. 5.13. We also will consider the bridge’s response to volume perturbations, either deliberate or inadvertent. The use of two different types of boundary conditions and the possibility of applying either of two profile-changing perturbations adds to the complexity of the analysis.

To investigate the different actuation schemes we will employ all the nondimensional parameters introduced in the standard configuration and pinned bridge cases:

\[
a = \sqrt{\frac{\pi h^3}{4V}}, \quad a^\circ = \frac{h}{D}, \quad U^\circ = \frac{V}{\pi D^2 h / 4}
\]

These three parameters are related to each other as:

\[
a = \frac{a^\circ}{\sqrt{U^\circ}}
\] (5.39)
Note that $a^\circ$ is the only parameter that is dependent only on the fixed geometry and independent of bridge volume. For a fixed value of $a^\circ$, parameters $a$ and $U^\circ$ may be used interchangeably.

5.7.2 Approach

Either a direct or an indirect approach may be employed to solve the boundary condition problem in this case. The direct approach requires two-dimensional root-finding and is therefore very time consuming. Here, we will derive the equations for the direct method and then explain how one may use numerical methods to get an approximate solution in a time efficient manner.

Denote the pinned radius and the contact radius on the active surface as $r_i$ and $r_u$, respectively. Pinned radius and nondimensional pinned radius, $\rho_i$, are related to each other as:

$$\kappa \rho_i = r_i$$

where $\kappa$ is the scale factor and may be found from equation above as:

$$\kappa = \frac{r_i}{\rho_i}$$

Nondimensional contact radius, $\rho_u$, may be found in terms of dimensionless pressure, $p$, and active contact angle, $\theta_u$, as:

$$\rho_u = P(\theta_u, p)$$

Bridge height and volume may be found as:

$$\kappa \zeta_j(\rho_i, \rho_u, p) = h$$
\[ V = \kappa^3 \nu_\epsilon(\rho_1, \rho_2, p) \]  \hfill (5.44)

Substitution of (5.41) and (5.42) into these equations yield:

\[
\left( \frac{r_i}{\rho} \right)^3 \nu_\epsilon(\rho_1, P(\theta_\epsilon, p), p) - V = 0
\]  \hfill (5.45)

\[
\left( \frac{r_i}{\rho} \right) \zeta_\epsilon(\rho_1, P(\theta_\epsilon, p), p) - h = 0
\]  \hfill (5.46)

Equations (5.45) and (5.46) are functions of two unknowns, \( p \) and \( \rho_1 \), and may be solved via two-dimensional root-finding. Because of the fairly wide range of dimensionless pressure, this solution method is very time consuming. An indirect method may alternatively be used to solve the boundary condition problem. To do so, a dimensionless pressure is selected from a set of discrete values. For this value of dimensionless pressure the inverse of the function \( \zeta(\rho) \), denoted, \( \hat{\rho}(\zeta) \) is constructed numerically. Denote the dimensionless distances of the neck/haunch (\( \zeta = 0 \)) from the pinned side and from the active side as \( \zeta_l \) and \( \zeta_u \) respectively. Then, Equation (5.43) may be written as:

\[ \zeta_l \pm \zeta_u = \pm h / \kappa \]  \hfill (5.47)

where the signs depend on the scenario number in Table 5.1. Equation (5.47) may be solved for \( \zeta_u \) to yield:

\[ \zeta_u = \pm h / \kappa \pm \zeta_l \]  \hfill (5.48)

Dimensionless radii, \( \rho_l \) and \( \rho_u \), may be found in terms of \( \zeta_l \) and \( \zeta_u \) as:

\[ \rho_l = \hat{\rho}(\zeta_l) \]  \hfill (5.49)

\[ \rho_u = \hat{\rho}(\zeta_u) \]  \hfill (5.50)

Substitution of (5.48) into (5.50) yields:
Equation (5.49) may be substituted into (5.41) to yield:

\[ r = \frac{r_i}{\hat{\rho}(\zeta_i)} \]  

(5.52)

Substitution for \( \kappa \) from Eqn. (5.52) into (5.51) and (5.44) yields:

\[ \hat{\rho}(\pm \frac{h}{r_i} \hat{\rho}(\zeta_i) \pm \zeta_i) \]  

(5.53)

Finally one may substitute for \( \rho_u \) and \( \rho_f \) from (5.49) and (5.53), respectively into (5.54) yielding:

\[ V = \left( \frac{r_i}{\hat{\rho}(\zeta_i)} \right)^3 u_j(\hat{\rho}_i, \hat{\rho}_u, p) \]  

(5.54)

(5.55)

Equation (5.55) is only in terms of one unknown, \( \zeta_i \), and may be solved via one-dimensional root-finding. Once the value of \( \zeta_i \) is found, one may find the contact angle \( \theta_u \) using the equation of slope in Table 2.1. With the profile parameters all determined, the capillary force may be calculated using Eqn. (2.30). Thus, for each selected value of nondimensional pressure (given \( h, r_i, \) and \( V \)) the other variables of interest may be determined. By sweeping through values of dimensionless pressure we can numerically determine the relationships between \( a^\phi \), \( f^\phi \), and \( \theta_e \).
5.7.3 Results and Discussion

Using the indirect approach discussed above, a design chart for case 6 was developed, see Figure 5.14. Since this problem has one more design variable than that for the standard configuration, it is necessary to fix one value of the configuration in order to solve. Here, we fix $U^\ominus = 1$ (i.e., volume is equal to that of a cylinder of radius $r_i$ and height $h$).

Figure 5.14 shows the cosine of contact angle on the pinned side as a function of cosine of active contact angle for various values of $a^\ominus$ when $U^\ominus = 1$. It may be observed that with change in the active contact angle, the passive contact angle behaves the same way for the values of pinned aspect ratio examined.

![Figure 5.14: Design chart for parallel plate configuration with bridge pinned on one side](image-url)
Example: Consider a bridge with values of height and volume equal to those typically used throughout this dissertation \((h = 100 \mu m\) and \(V = 7.85 \times 10^7 \mu m^3\)) and a pinned radius of \(r_i = 500 \mu m\). For these values, \(U^0 = 1\), and \(a^0 = 0.1\). If electrowetting is used to control the capillary force, the effectiveness \(\Delta F / \Delta \cos(\theta_c)\) would be \(864 \mu N\). This is about 80% of the effectiveness calculated for the standard configuration (Case 1) and more than 1.6 times the effectiveness of Case 2 where the contact angle is fixed on the passive plate, rather than pinned.

Volume Perturbations: Now we consider the performance if volume perturbations (either deliberate or inadvertent) alter the bridge shape. Figure 5.16 shows the change in nondimensional force, dimensionless pressure and contact angle of the pinned side for several values of fixed contact angles on the non-pinned side. From the comparison of these plots with those of the case where the bridge is pinned on both sides, it may be
observed that the maximum change in capillary force produced with volume perturbation is about half that of the both-sides-pinned case.

Figure 5.16: (a) Nondimensional force, (b) dimensionless pressure, (c) pinned contact angle as a function pinned dimensionless volume for \( \theta^0 = 0.1 \)
5.8 Cavity Configuration

5.8.1 Introduction

In this section we will consider an actuator consisting of a passive surface and an opposing active surface which contains a cavity into which the liquid bridge wets, see Figure 5.17. The cavity contains electrodes, each covered by a dielectric layer. As we will discuss, for reasons of stability both surfaces have a hydrophobic layer with the surface energy tailored as a function of radius.

![Figure 5.17: The cavity configuration](image)

5.8.2 General Stability Analysis

We will first examine under what condition the bridge will not spontaneously de-wet either surface. This is a question of stability. Let \( r, V \) and \( E \) denote the radius, volume and potential energy of various sections of the bridge shown in Figure 5.17, respectively. We denote the liquid/solid and gas/liquid surface of each section by \( S_\text{ls} \) and \( S_\text{gl} \) respectively. Indices for these parameters will denote whether the upper section (index=1) or lower section (index=2) is considered. Since the liquid is incompressible, the total volume of the bridge is constant (i.e. \( V_1 + V_2 = V \)). For stability analysis, let the
spacing between the surfaces be fixed. Taking a variation of the total volume with respect to the contact line radii yields:

\[
\frac{\partial V_1}{\partial r_1} dr_1 + \frac{\partial V_2}{\partial r_2} dr_2 = 0
\]

(5.56)

\[
\Rightarrow dr_2 = \frac{-\frac{\partial V_1}{\partial r_1}}{\frac{\partial V_2}{\partial r_2}} dr_1
\]

(5.57)

For equilibrium, a necessary condition is that the variation of the total potential energy, 

\[ E = E_1 + E_2 \]

is zero:

\[
\frac{\partial E}{\partial r_1} dr_1 + \frac{\partial E_2}{\partial r_2} dr_2 = 0
\]

(5.58)

Substitution of the expression for \( dr_2 \), Eqn. (5.57), yields:

\[
\frac{\partial E_1}{\partial r_1} dr_1 + \frac{\partial E_2}{\partial r_2} \frac{-\frac{\partial V_1}{\partial r_1}}{\frac{\partial V_2}{\partial r_2}} dr_1 = 0
\]

(5.59)

\[
\Rightarrow \frac{\partial E_1}{\partial V_1} = \frac{\partial E_2}{\partial V_2}
\]

(5.60)

This is a necessary condition for equilibrium. A sufficient condition for stability of the equilibrium is obtained by examining the second variation of the energy:

\[
\frac{\partial^2 E}{\partial r_1^2} dr_1^2 + \frac{\partial^2 E_2}{\partial r_2^2} dr_2^2 > 0
\]

(5.61)

Using Eqn. (5.60), we obtain:
\[
\frac{\partial^2 E_1}{\partial r_1^2} + \frac{\partial^2 E_2}{\partial r_2^2} \left( \frac{\partial V_1}{\partial r_1} \right)^2 > 0 \quad (5.62)
\]

or

\[
\frac{\partial^2 E_1}{\partial r_1^2} + \frac{\partial^2 E_2}{\partial r_2^2} \left( \frac{\partial V_1}{\partial r_1} \right)^2 > 0 \quad (5.63)
\]

As is evident, the denominators of both terms are always positive. However, the numerators may not necessarily be so. To investigate, we first examine the energy for the upper section:

\[
E_1 = -\sigma_{g_1} \cos(\theta_1) S_{b_1} + \sigma_{g_1} S_{g_1} \quad (5.64)
\]

Note that $S_{b_1}$ is for the most part quadratic in $r_1$ while $S_{g_1}$ is effectively a linear function of $r_1$. To begin, we will assume that contact angle is independent of radius.

Therefore,

\[
\frac{\partial^2 E_1}{\partial r_1^2} \approx -\sigma_{g_1} \cos(\theta_1) \frac{\partial^2 S_{b_1}}{\partial r_1^2} \quad (5.65)
\]

Similarly, for the lower section we have:

\[
\frac{\partial^2 E_2}{\partial r_2^2} \approx -\sigma_{g_2} \cos(\theta_2) \frac{\partial^2 S_{b_2}}{\partial r_2^2} \quad (5.66)
\]

The stability condition of Eqn. (5.61) then becomes:

\[
\frac{-\sigma_{g_1} \cos(\theta_1) \frac{\partial^2 S_{b_1}}{\partial r_1^2} + \sigma_{g_2} \cos(\theta_2) \frac{\partial^2 S_{b_2}}{\partial r_2^2}}{\left( \frac{\partial V_1}{\partial r_1} \right)^2 + \left( \frac{\partial V_2}{\partial r_2} \right)^2} > 0 \quad (5.67)
\]
For flat surfaces normal to the bridge axis, \( \frac{\partial^2 S_{b1}}{\partial r_1^2} \) and \( \frac{\partial^2 S_{b2}}{\partial r_2^2} \) will be positive in general.

In this case, the equilibrium obtained cannot be stable if both sides are hydrophilic (\( \cos(\theta) > 0 \)). This presents a problem for actuation since the cavity surface will achieve strong hydrophilicity under electrowetting, while \( \cos(\theta) \) for the passive surface will only be slightly negative (at best). One approach to remedy this is to shape the upper surface so as to cause \( \frac{\partial^2 S_{b1}}{\partial r_1^2} \) to be negative. Another approach, examined below, is to vary the contact angle \( \theta \) (equivalently, \( \cos(\theta) \)) as a function of contact line radius on a flat surface.

Denoting the cosine of contact angle as a function of a radius as \( C(r) \), the second derivative of energy with respect to radius would be:

\[
\frac{\partial^2 E_1}{\partial r_1^2} \approx -\sigma_g \left[ C_1(r_1) \frac{\partial^2 S_{b1}}{\partial r_1^2} + 2 \frac{dC_1}{dr}(r_1) \frac{\partial S_{b1}}{\partial r_1} + 2 \frac{d^2C_1}{dr^2}(r_1) S_{b1} \right] \tag{5.68}
\]

We may obtain a similar equation for the lower part as well. This shows that the contact angle gradient functions \( C_1(r) \) and \( C_2(r) \) may be chosen to have negative gradient with respect to radius (\( \frac{dC_1}{dr} \) and \( \frac{dC_2}{dr} \)) to achieve a positive gradient of energy \( \frac{\partial^2 E_1}{\partial r_1^2} \) and \( \frac{\partial^2 E_2}{\partial r_2^2} \) and thus a positive \( \partial^2 E \), which will result in a stable equilibrium.
5.8.3 Equilibrium Shape

To find the equilibrium shape of the bridge for the cavity configuration, two coupled boundary condition problems must be solved, those corresponding to the equations for the upper and lower sections of the bridge. The known parameters in the overall mathematical problem are the contact angle gradient function for each part, $C_1(r)$ and $C_2(r)$, the pinned radius of the upper part $r_{l1}$, the height of each bridge section, $h_1$ and $h_2$, and the total volume of the liquid $V$. The unknowns for the overall problem would be volume of each section, $V_1$ and $V_2$, dimensionless pressure of each section, $p_1$ and $p_2$, and the dimensionless contact radii for each part.

Each of these boundary condition problems has been investigated independently in the earlier sections of this chapter. To solve the problem as a whole, we will use the method illustrated in Figure 5.18.

The two sections of the configuration are connected to each other by flow of liquid. Since gravitational forces are negligible at this scale, we may assume a constant pressure throughout the liquid in lower and upper sections of the configuration. From the definition of the dimensionless pressure we have

$$P_c = \frac{2\sigma_{lg} p}{r_0} = 2\sigma_{lg} \kappa \text{Sign}[p]$$

(5.69)

where $\kappa$ denotes the scaling factor defined earlier. Equating the expression above for the two sections of the configuration yields:

$$\kappa_1 = \kappa_2 \text{ and } \text{Sign}[p_1] = \text{Sign}[p_2]$$

(5.70)
The analysis problem may be split into two parts, each with certain “inputs” and “outputs”. For the lower section scale factor $\kappa_2$ and change in contact angle $\Delta C$ are considered as inputs. The change in contact angle represents the effect of electrowetting on the surfaces of the lower section.

For the upper section the only input is the scale factor $\kappa_2$. Each section of the problem requires root finding. To simplify the solution one may avoid root finding by employing interpolation functions. The upper section may be viewed as a bridge problem with a pinned boundary condition on one side and volume perturbations. This case has been investigated in an earlier section of this chapter. Solving this boundary condition problem for a range of dimensionless pressures, we may construct interpolation functions of $V_1(\kappa_1)$ and $p_1(\kappa_1)$. The lower section is a parallel plate case with a gradient of contact angle. Interpolation functions $V_2(\kappa_2, \Delta C)$ and $p_2(\kappa_2, \Delta C)$ for certain values of $\Delta C$ may also be constructed. The equations used to numerically solve and so define these interpolation functions are presented in Table 5.4.

Figure 5.18: Method used to solve the cavity’s coupled boundary condition problems
Having these interpolation functions one may find the appropriate value of $\kappa^* = \kappa_1 = \kappa_2$ for each value of $\Delta C$ by solving the numerical problem of $V_1(\kappa^*) + V_2(\kappa^*, \Delta C) = V$.

Dimensionless pressure for each part may be computed as $p_1(\kappa^*)$ and $p_2(\kappa^*, \Delta C)$. It is important to note that because the force is delivered by the upper section, which is the same as a capillary bridge with a pinned condition on one side, the cavity case will not be capable of producing greater total force than the parallel plate case with one pinned condition and volume perturbations. Nevertheless, the cavity configuration is capable of producing the same force at much lower voltage as the following example illustrates.

Table 5.4: Equations used to construct the interpolation functions for cavity problem

<table>
<thead>
<tr>
<th>Upper Section</th>
</tr>
</thead>
<tbody>
<tr>
<td>Known: $C_1(r), r_{i1}, h_1$</td>
</tr>
<tr>
<td>Unknown: $\kappa_1, \rho_{h1}, \rho_{l1}$</td>
</tr>
<tr>
<td>$\kappa_1 \rho_{l1} = r_{i1}$</td>
</tr>
<tr>
<td>$C_1(\kappa_1 \rho_{h1}, p_1) = \cos \Phi(\rho_{h1}, \rho_{l1})$</td>
</tr>
<tr>
<td>$\kappa_1 [\zeta(\rho_{h1}, p_1) - \zeta(\rho_{l1}, p_1)] = h_1$</td>
</tr>
<tr>
<td>Solve $[\zeta(\rho_{h1}, p_1) - \zeta(\rho_{l1}, p_1) = h_1 \rho_{l1} / r_{i1}, \rho_{h1} / \rho_{l1}, \rho_{l1}, \rho_{h1}]$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Lower Section</th>
</tr>
</thead>
<tbody>
<tr>
<td>Known: $C_2(r), \theta_{h}, \rho_{c}, p_2, h_2$</td>
</tr>
<tr>
<td>Unknown: $\rho_{c}, \kappa_2$</td>
</tr>
<tr>
<td>$h_2 = \kappa_2 \zeta(\rho_{c}, p_2)$</td>
</tr>
<tr>
<td>$C_2(\kappa_2 \rho_{c}) + \Delta C = \cos \Phi(\rho_{c}, p_2)$</td>
</tr>
<tr>
<td>Solve $[\rho_{c} / \rho_{c} / \rho_{c} / \rho_{c}]$</td>
</tr>
</tbody>
</table>
**Example:** We consider an example of cavity configuration to understand the method of solution and the specific properties of the configuration. The numerical values used here are chosen very close to the typical values used for standard configuration. The bridge height of the upper and lower section are considered to be $50 \mu m$ and $20 \mu m$, respectively.

For the top surface of the upper section, a gradient of contact angle with $50^\circ$ contact angle region of radius $450 \mu m$ at the center, a transition region of radial width $100 \mu m$, and $80^\circ$ contact angle on the exterior region is employed. Both plates on the bottom section carry an energy gradient with hydrophilic center (contact angle $80^\circ$) of radius $550 \mu m$, a transition region of radial width $300 \mu m$, and a hydrophobic exterior region (contact angle $120^\circ$). The volume of the bridge is considered to be $9.5 \times 10^7 \mu m^3$. In the case of no electrowetting ($\Delta C = 0$) the bridge radius in the bottom section was $600 \mu m$. This will be increased to $690 \mu m$ with a change in contact angle cosine of $\Delta C = 0.3$. This change will result in transfer of $0.74 \times 10^7 \mu m^3$, close to 8% of the total volume, from the upper section to the lower section and a change in the attractive force from $740 \mu N$ under no applied voltage to $840 \mu N$ under change of contact angle cosine of $\Delta C = 0.3$. Further electrowetting will result in very low pressure inside the liquid and consequently spontaneous wetting of the entire liquid volume into the lower section. This demonstrates that large changes in upper section volume can occur with small applied voltage, and hence significant forces. But it also shows that this configuration is very sensitive and that this sensitivity is associated with destabilization of the configuration with increased electrowetting.
5.9 Application of Membrane in Capillary Force Actuators

Up to this point, we have considered the use of capillary force actuation for the movement of one rigid surface toward or away from another. Now, we will consider the use of this actuation principle for the deflection of a flexible membrane. Researchers in micro systems have studied the employment of micro membranes in pressure sensors and other acoustic devices. Analytical investigation has been conducted and theories for both small deflection and large deflection have been proposed. The small deflection theory is appropriate when the deflection is less than 1/5 of the diaphragm thickness. The theory of large deflections loses its accuracy for deflections greater than 5 times the membrane thickness. In both theories, deflection $w$ of a circular membrane is given by

$$w(r) = d \left[ 1 - \left( \frac{r}{r_{mem}} \right)^2 \right]^2$$

(5.71)

where $r, r_{mem}$ are the radial coordinate and membrane radius, respectively, and the maximum deflection, $d$, is governed by expressions in the small and large deflection theories. In small deflection theory $d$ is given by

$$d = \frac{Pa^4}{64\Gamma}$$

(5.72)

where $P$ is the uniform pressure applied to the membrane and $\Gamma$ is a measure of stiffness called flexural rigidity

$$\Gamma = \frac{E_{mem}t_{mem}^3}{12(1-\nu^2)}$$

(5.73)

where $E_{mem}, t_{mem},$ and $\nu$ are Young’s modulus, membrane thickness, and Poisson’s ratio, respectively. For large deflections $d$ may be obtained as:
where $\eta$ and $\mu$ are given as:

$$
\eta = 14 \frac{4t_{mem}^3 + 3r_{mem}^2 \varepsilon_{mem} (1 + \nu)}{(1 + \nu)(23 - 9\nu)} \\
\mu = \frac{-7P r_{mem}^4 t_{mem}^2}{8\Gamma (1 + \nu)(23 - 9\nu)}
$$

where $\varepsilon_{mem}$ is the built-in residual strain.

Independent of the theory used, the volume displaced by a deflected membrane may be obtained by integration of the deflection equation over the whole plate.

$$
\Delta V = \int_0^{r_{mem}} d \left[ 1 - \left( \frac{r}{r_{mem}} \right)^2 \right]^2 (2\pi r) dr = d \frac{\pi r_{mem}^2}{3}
$$

A comparison of the standard configuration to that with membrane yields some interesting insights.

Figure 5.19: CFA without and with application of membrane

In the standard configuration, as one surface approaches the other $h$ will become smaller, decreasing the capillary pressure even when contact angle is constant:

$$
P_c = \sigma \left( \frac{1}{r} - \frac{2 \cos(\theta_c)}{h} \right)
$$

This decrease in pressure tends to cause the surface to move further toward the other.

Either equilibrium will be reached (due to a mechanical restoring force) or “snap-in”
instability will occur. For the membrane case, \( h_0 \) does not change with deflection of the membrane. The membrane motion will increase \( r \), hence making \( 1/r \) smaller. While this phenomenon tends to decrease capillary pressure and increase membrane deflection, it is a small effect. (The same effect happens in the standard configuration.)

We may analyze this phenomenon. From the earlier calculation we have:

\[
dV = \frac{\pi r_{\text{mem}}^2}{3} dy
\]  

(5.78)

where \( dy \) is an infinitesimal change in center displacement. The volume covered as fluid moves out is

\[
dV = 2 \pi r h dr
\]  

(5.79)

Equating the two expressions above yields:

\[
\frac{dr}{dy} = \frac{r_{\text{mem}}^2}{6rh}
\]  

(5.80)

Integrating both sides we get:

\[
\frac{\pi r_{\text{mem}}^2}{3} (y - y_0) = 2 \pi h \int_{r_0}^r w dw = \pi h (r^2 - r_0^2)
\]

or

\[
r^2 = r_0^2 + \frac{r_{\text{mem}}^2}{3} \frac{(y - y_0)}{h}
\]  

(5.81)

Since \( r_0 > r_{\text{mem}} \), the second term is significant only if the deflection is nearly equal to \( h \).
5.10 Non-Parallel Plates

We now examine a simple variation from the conventional geometry, the case of non-parallel plates. This case is of interest for a number of applications where the force exerted by the actuator results in a rotation of one plate. For example, consider a switch to be employed in RF MEMS. Actuation causes the deflection of a cantilever beam at one end.

For this case, the angle of deviation from parallel will be called the tilt angle and denoted $\alpha$ (see Figure 5.20). The effect of tilt angle upon the bridge will be investigated for both hydrophobic and hydrophilic surfaces.

Numerical investigation using Surface Evolver shows a tendency of the bridge to move toward the interstice for hydrophilic plates and to move away for hydrophobic plates. An analytical investigation of this behavior may be conducted by considering the effect of bridge lateral motion on its interfacial energy. The total interfacial energy is given by:

$$E = \sigma_{gl} \left( S_{gl} - \cos(\theta_0) S_{ls} \right) + \sigma_{gs} S_s$$

(5.82)
Considering the bridge volume to be constant, scaling of the different terms in the equation may be performed. Here, we assume the bridge to be a cylindrical segment with radius \( r_{cyl} \) and center height \( h \). The volume of the bridge is:

\[
V = \pi r_{cyl}^2 h \rightarrow r_{cyl} = \sqrt[3]{\frac{V}{\pi h}} 
\]  
(5.83)

The surface areas may be found:

\[
S_{ls} = \pi r_{cyl}^2 \left( 1 + \frac{1}{\cos(\alpha)} \right) \quad (5.84)
\]

\[
S_{gl} = \pi r_{cyl} h 
\]  
(5.85)

We note that the third term of the energy \( \sigma_{gs} S_s \) is independent from the configuration parameters and is constant. Substitution yields:

\[
E = \sigma_{gl} \left( \pi r_{cyl} h - \cos(\theta_0) \pi r_{cyl}^2 \left( 1 + \frac{1}{\cos(\alpha)} \right) \right) + \sigma_{gs} S_s 
\]  
(5.86)

Using the expression for the radius in terms of the bridge volume and height yields:

\[
E = \sigma_{gl} \left( \sqrt{\pi V h} - \cos(\theta_0) \left( 1 + \frac{1}{\cos(\alpha)} \right) \frac{V}{h} \right) + \sigma_{gs} S_s 
\]  
(5.87)

Taking a derivative with respect to bridge height yields:

\[
\frac{1}{\sigma_{gs}} \frac{dE}{dh} = \frac{1}{2} \sqrt{\pi} \sqrt{\frac{V}{h}} + \cos(\theta_0) \left( 1 + \frac{1}{\cos(\alpha)} \right) \frac{V}{h^2} 
\]  
(5.88)

The important factor in determining the behavior of the bridge is the sign of \( \frac{dE}{dh} \). If \( \frac{dE}{dh} \) is positive, the bridge’s energy is reduced by moving into the interstice (smaller \( h \)). If \( \frac{dE}{dh} \) is negative, the bridge’s movement will be away from the interstice. Eqn. (5.88) may be rearranged to yield:
\[
\frac{1}{
\frac{\sigma_{gl}}{\sigma}\ \frac{dE}{dh}} = \sqrt{\frac{\pi V}{4 h}} \left[ 1 + \cos(\theta_o) \left( 1 + \frac{1}{\cos(\alpha)} \right) \left( \frac{1}{a} \right) \right]
\]

\[
= \sqrt{\frac{\pi V}{4 h}} \left( \frac{1}{a \cos(\alpha)} \right) \left[ a \cos(\alpha) + \cos(\theta_o)(\cos(\alpha) + 1) \right]
\]

\[
= \sqrt{\frac{\pi V}{4 h}} \left( \frac{1}{a \cos(\alpha)} \right) \left[ \cos(\theta_o) + \cos(\alpha)(a + \cos(\theta_o)) \right]
\]

(5.89)

Note that the term outside the brackets is always positive. We will assume that \( \alpha \) is small and therefore \( \cos(\alpha) \approx 1 \). We may conclude that for contact angles \( \theta < 90^\circ \) the sum of the two terms inside the brackets and hence \( \frac{dE}{dh} \) is positive. For contact angles \( \theta > \cos^{-1}(a/2) \), which is near \( 90^\circ \) for low aspect ratio bridges, the sum will be negative, hence \( \frac{dE}{dh} < 0 \). This result confirms the tendency of inward movement for bridges sandwiched between hydrophilic plates and outward movement for hydrophobic plates.

Due to this transverse instability of the bridge in such a configuration, we will investigate next the introduction of a gradient in surface energy (contact angle) upon the plate surfaces.

5.11 Parallel Plate Configuration with Non-Axisymmetric Surface Energy Gradient

In this section we will evaluate the effect of a non-axisymmetric gradient in surface energy on force production by a liquid bridge. We will also examine the consequences this has for bridge shape and the contributions of surface tension and capillary pressure terms to total force. Since circular symmetry is lost, we must employ a numerical approach.
For this investigation the spacing between plates and the volume of the drop are chosen to be values equal to those used throughout this dissertation \( (h = 100 \mu m \) and \( V = 7.85 \times 10^7 \mu m^3) \). The contact angle on the surfaces ranges from 60° to 120°. The analysis starts with a linear gradient surrounding a wide hydrophilic central region and proceeds incrementally toward one where the hydrophilic region is narrow (Figure 5.21). The equilibrium diameter of the contact line for this volume of bridge when placed between uniformly hydrophilic surfaces with contact angle 60° was previously found to be 1018 \( \mu m \).

The starting point for analysis is a gradient along one axis with the diameter of hydrophilic region set to 1000 \( \mu m \). The gradient from hydrophilic (60°) to hydrophobic (120°) occurs over 100 \( \mu m \). In this analysis, the width of the hydrophilic region is reduced while the width of the transition region is held constant. For this study the effect of reducing the width of the hydrophilic region from 1000 \( \mu m \) to 500 \( \mu m \) is investigated.
Figure 5.22 illustrates the surface energy gradient function \( \cos(\theta_o) \) versus distance from the center of the plate for the case with a hydrophilic region half-width of 500 \( \mu \text{m} \).

As the width of the hydrophilic region is decreased, the wetted area will vary from a circle to a narrow oval as the Surface Evolver results shown in Figure 5.23 demonstrate.

Figure 5.22: Illustration of contact angle gradient function employed.

Figure 5.23: Gradual change of wetted area shape with the reduction of hydrophilic region width. The three shapes correspond to the three surfaces shown in Fig. 5.21.
The surface tension contribution to the force $F_{\sigma}^{SE}$, capillary pressure contribution to the force $F_{p}^{SE}$ and the total force $F_{c}^{SE}$ may be determined from Surface Evolver data via:

$$F_{p}^{SE} = S_{ls}^{SE} P_{c}^{SE} \sigma_{gl}$$  \hspace{1cm} (5.99)

$$F_{\sigma}^{SE} = -\int_{L^{SE}}^{\sigma_{gl}} \sin(\phi) ds$$ \hspace{1cm} (5.100)

$$F_{c}^{SE} = F_{p}^{SE} + F_{\sigma}^{SE} = [S_{ls}^{SE} P_{c}^{SE} - \int_{L^{SE}} \sin(\phi) ds] \sigma_{gl}$$ \hspace{1cm} (5.101)

where $S_{ls}^{SE}$ is the wetted area, $L^{SE}$ represents the perimeter of the wetted area, $\phi$ is the contact angle of the drop on the plate, and $P_{c}^{SE}$ is the capillary pressure reported by Surface Evolver which is related to dimensionless pressure as:

$$P_{c}^{SE} = \frac{P_{c}}{\sigma_{gl}}$$ \hspace{1cm} (5.102)

Figure 5.24 shows these force components and total force as a function of the width of the hydrophilic region. These results illustrate the force produced by the bridge is weakly dependent on the shape of the bridge wetted area. This suggests that surface energy gradients could be used for stabilization of a bridge under tilting without significantly affecting device performance. It also suggests that the performance of CFA will not be highly sensitive to the precise geometry of any engineered surface energy gradient.
Figure 5.24: (a) The surface tension force contribution, (b) capillary pressure force contribution, and (c) total force as a function of width of hydrophilic region.
5.12 Non-parallel Plates with Surface Energy Gradient

As the results of analytical and numerical investigations in Section 5.10 revealed, a capillary bridge between two plates with a tilt angle will not be laterally stable. It will tend to move toward (away) the interstice for a hydrophilic (hydrophobic) surface. Furthermore, in Section 5.11 it was shown that a surface energy gradient along one direction on the surfaces may alter the wetted area but will not significantly change the resulting capillary force. In this section, we will consider two plates with a tilt angle $\alpha$ and introduce a surface energy gradient on the lower plate to provide lateral stability.

Because the configuration is not axisymmetric, a numerical method must be used. Surface Evolver is employed and the configuration is very similar to that of Section 5.10 (see Fig. 5.25). Dimensions are chosen to resemble the application of CFAs in micro cantilevers for RF MEMS. For this analysis, the distance between the centers of the plates is assumed to stay constant and is equal to $1.5 \mu m$. The tangent of the tilt angle is varied between 0 and 0.04 corresponding to change in tilt angle from $0^\circ$ to $2.3^\circ$, respectively. The volume of the bridge is considered to be $678.58 \mu m^3$ (equal to the volume of a cylindrical bridge with radius $12 \mu m$ and height $1.5 \mu m$). The initial (without electrowetting) contact angle on the top plate is considered to be $100^\circ$. A surface energy gradient on the bottom plate is considered consisting of a central region with width of $20 \mu m$ and a contact angle of $100^\circ$, a gradient region of length $10 \mu m$ on each side of the center and surrounding hydrophobic region with contact angle $140^\circ$. 
First we will consider the surface tension contribution, capillary pressure contribution, and the total amount of force produced without electrowetting for different values of tilt angle, see Figure 5.26. We note that the change in components and in the total amount of capillary force with increasing tilt is not very significant.

One may consider the effect of electrowetting as a uniform shift in the cosine of contact angle throughout both surfaces. Figure 5.27 shows the capillary force components as well as the total force as a function of change in contact angle cosine, $n$, for three values of tilt angle. The graphs in Fig. 5.27 indicate that tilt angle does not have a strong effect on force output for this example. Therefore, we expect that a gradient of surface energy may be used as an inexpensive and effective tool to laterally stabilize the bridge in the tilt case, allowing electrowetting to be employed.
Figure 5.26: (a) The surface tension force contribution, (b) capillary pressure force contribution, and (c) total force as a function of tangent of tilt angle.
Figure 5.27: (a) The surface tension force contribution, (b) capillary pressure force contribution, and (c) total force as a function of change in contact angle cosine.
6 Conclusions and Future Work

6.1 Conclusions:

In this dissertation, the physics and design principles of capillary force actuators are examined. Semi-analytical equations governing the capillary bridge profile under no applied voltage are given and methods to find the bridge shape are presented. An electromechanical analysis of the device under electric potential is performed and an energy minimization approach is used to determine the bridge shape under electric field. Once the bridge shape is known, two different approaches are introduced to determine the capillary force produced. Using the methods introduced it is shown that CFA is capable of producing a force more than 10 times greater than that achievable with electrostatic actuators of comparable size. In addition, numerical methods to find the bridge shape and capillary force are introduced, which are capable of modeling more complex (non-axisymmetric) configurations of CFAs.

An analytical approximation of the capillary force for the standard configuration is developed based on the principle of virtual work and design limits and actuator optimization are investigated. The impact of the contact angle saturation and dielectric breakdown on device performance is considered. It is shown that maximum achievable force is independent of liquid surface tension when the applied voltage is fixed. It is further argued that, once material properties are given, an optimal thickness for the dielectric layer may be computed such that maximum force is achieved before dielectric breakdown or contact angle saturation occurs. The scaling of the actuator force produced
is considered and it is shown that force produced by CFA scales as the first power of linear dimension. This scaling relationship is shown to be more favorable than electrostatic or electromagnetic for microdevices.

The stability of axisymmetric capillary bridges is investigated. First, the bridge profiles are categorized as either longitudinally symmetric (equal contact angles) or non-symmetric (unequal contact angles). The existing work for symmetric bridge profiles is transformed into dimensionless parameters (aspect ratio and dimensionless pressure). In the case of non-symmetric bridges, the stability limit in terms of aspect ratio is computed for pairs of contact angles. The lowest stability limit over the range of contact angles considered is determined and it is shown that aspect ratios corresponding to the typical values of configuration parameters considered for CFA will always fall in the stable region. Furthermore, numerical methods for examining the stability of a bridge in more complicated configurations are introduced.

Nine alternative configurations of CFAs are considered and the force output of each configuration is computed and compared to that of the standard configuration. Six cases of boundary conditions for the axisymmetric case (including bridges with gradient of contact angle, fixed contact angle, or pinned radius on one side) are investigated. It is shown that the standard configuration is capable of producing a larger range of force than any of the alternative configurations considered. However, the rate of change in force may be further enhanced in the standard configuration by employing a surface energy gradient. In the case of only one active surface, it is shown that pinning the passive side
would result in better performance than a fixed contact angle on the passive surface. It is also shown that altering the fixed contact angle on the passive side only shifts the achievable force interval and does not affect the maximum change in force. Three cases of non-axisymmetric configurations (non-parallel plates, non-axisymmetric gradient, and the combination of these two) are considered as well. It is shown that the use of a non-axisymmetric gradient will change the shape of the wetted area, but will not affect the amount of force produced significantly. Presence of tilt angle between the plates is shown to cause lateral instability of the bridge, which may be prevented by employing a surface energy gradient. In this case, the amount of force produced is shown to be essentially constant when the tilt angle is kept relatively small. A cavity configuration is also considered and shown to improve the actuator effectiveness, but reduce its stability. The application of CFA to the deflection of a micro membrane is also studied.

6.2 Future Work

Electrowetting tests would be the first step in the experimental phase of this work. Low voltage electrowetting reported by different groups may be replicated by using very thin layers of dielectric material and a hydrophobic coating. Experiments may be developed to measure the capillary force exerted directly or to witness it indirectly when the theoretical force would be sufficient to cause a measurable motion (such as bending a beam or deflecting a membrane). An experimental exploration of various dielectric layers and conductive liquids would be useful for further device development.
Contact angle saturation and hysteresis and their effect on force production may be
studied by conducting experiments with different coatings of the dielectric layer. An
investigation of the effect of high intensity electric field at the three phase contact line on
the molecular structure of dielectric and potential breakdown would also be desirable.
These studies would strongly depend on the chemical and structural properties of the
materials employed and would require a series of experiments.

Experiments for the different boundary conditions (such as passive surface with pinned
contact line radius or fixed contact angle) should be conducted and the practical
advantages or disadvantages of each case investigated. Different techniques to pin the
bridge or produce surface energy gradient should also be explored. Finally, investigations
should be conducted to examine the dynamics of the device, including fluid dynamics of
the liquid bridge and that of the electric field. This would be particularly important when
considering applications where high frequency actuation is required.
References


Appendix A: Evaporation

Considering the small volume of the liquid in the device it is reasonable to suspect that evaporation may have a deleterious effect on operation. In this section, a detailed analysis of evaporation is presented and several conclusions are drawn. While evaporation is inherently a dynamic phenomenon, for given initial and boundary conditions we may examine the static problem of determining the steady state that is reached. The configuration chosen for this analysis, shown in Figure A.1, is a droplet in a closed chamber containing air and potentially liquid vapor.

The partial pressure and the volume of the air inside the container are denoted by \( P_a \) and \( V_a \) respectively. Variables \( P_v \) and \( V_v \) denote the partial pressure and the volume of the liquid vapor. A Dalton model is applied for the mixture of liquid vapor and air inside the container. Thus:

\[
P_g = P_a + P_v \tag{A.1}
\]

\[
V_g = V_a = V_v \tag{A.2}
\]
where $P_g$ and $V_g$ represent the total pressure and the total volume of the gas inside the container. Due to the small volume of the droplet, the pressure gradient inside the liquid caused by the gravity may be ignored and the liquid may be assumed to have a uniform internal pressure, denoted $P_l$. The volume of the liquid is denoted $V_l$.

The volume of the drop is assumed to be a function of a characteristic dimension (radius) $R$:

$$V_l = g(R)$$  \hspace{1cm} (A.3)

The liquid-gas surface tension is balanced by a pressure difference between the liquid and the surrounding gas. The pressure difference is related to the characteristic dimension via Laplace’s Eqn. Functionally, this may be represented as:

$$P_l - P_g = f(R)$$  \hspace{1cm} (A.4)

Functions $g()$ and $f()$ are dependent on the droplet geometry. We will return to this aspect shortly. The ideal gas assumption is used for both the air and the liquid vapor inside the container, hence:

$$P_a V_g = m_a \mathcal{R}_a T_0$$  \hspace{1cm} (A.5)

and

$$m_v = \frac{P_v V_g}{T_0 \mathcal{R}_v}$$  \hspace{1cm} (A.6)

where $m_a$ and $m_v$ denote the mass of air and vapor in the chamber and the temperature of the system is denoted by $T_0$. Indices for the parameters introduced above other than temperature will denote whether the initial value (index=1) or final value (index=2) is
considered. The initial state for this analysis is the absence of liquid vapor in the container. Therefore,

\[ P_{v1} = 0 \text{ and } P_{g1} = P_{a1} \]

where the final state is reached the vapor inside the container and the liquid are in equilibrium (i.e. the rate of evaporation and condensation are the same). In this condition there is no further change in either the volume of the liquid or the volume of the vapor inside the container. The liquid and vapor pressures will be related by the equation:

\[ P_{v2} = c P_{l2} \]

where \( c \) is a constant determined by the fluids employed and the temperature of the device. An investigation of evaporation in [1] showed that typical values for \( c \) are 0.85 to 1.0. Two more relationships between the variables are necessary for this analysis: conservation of liquid mass and conservation of the total volume in the container. With these, the analysis problem may be formulated as shown in Table A.1.

<table>
<thead>
<tr>
<th></th>
<th>State 1</th>
<th>State 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Surface Tension</td>
<td>( P_{l1} - P_{a1} = f(R_1) )</td>
<td>( P_{l2} - (P_{a2} + P_{v2}) = f(R_2) )</td>
</tr>
<tr>
<td>Volume of drop</td>
<td>( V_{l1} = g(R_1) )</td>
<td>( V_{l2} = g(R_2) )</td>
</tr>
<tr>
<td>Ideal gas assumption for air</td>
<td>( P_{a1} V_{g1} = P_{a2} V_{g2} )</td>
<td></td>
</tr>
<tr>
<td>Constant total volume</td>
<td></td>
<td>( V_{g1} + V_{l1} = V_{g2} + V_{l2} )</td>
</tr>
<tr>
<td>Constant total mass of liquid</td>
<td></td>
<td>( \rho_w (V_{l1} - V_{l2}) = \frac{P_{v2} V_{g2}}{T_0 \mathcal{R}_w} )</td>
</tr>
<tr>
<td>Equilibrium Condition</td>
<td></td>
<td>( P_{v2} = c P_{l2} )</td>
</tr>
</tbody>
</table>
Combining the equations together to eliminate variables and simplifying, the functional form of the analysis problem is:

\[
\frac{1}{(c-1) \rho w} \mathcal{R}_w T - f(R_2) \left( g(R_1) - g(R_2) \right) - (P_{e1} + f(R_2)) V_{g1} = 0 \quad \text{(A.7)}
\]

with a single unknown \( R_2 \).

We now consider the functions \( f \) and \( g \) introduced earlier. These are given for three cases in Table A.2.

### Table A.2: Geometry dependent functions for different cases of capillary bridges

<table>
<thead>
<tr>
<th>Case</th>
<th>( f(R) )</th>
<th>( g(R) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sessile Drop Contact Angle ( \theta )</td>
<td>( \frac{2\sigma}{R} )</td>
<td>( \pi \left( \frac{2/3}{4} - \frac{3\cos \theta}{12} \right) R^3 )</td>
</tr>
<tr>
<td>Cylindrical Bridge Bridge Height ( h )</td>
<td>( \frac{\sigma}{R} )</td>
<td>( \pi h R^2 )</td>
</tr>
<tr>
<td>Bridge- Surface of Revolution of a Circular Arc Contact Angle ( \theta ) and Bridge Height ( h )</td>
<td>( \sigma \left( \frac{1}{R} - \frac{2}{h} \cos \theta \right) )</td>
<td>( \pi h R^2 + \frac{\pi}{2 \cos \theta} h^2 R )</td>
</tr>
</tbody>
</table>

Some typical values for the material and configuration properties (see Table A.3) have been used in numerical solution of this problem. Each property has been varied individually to investigate the sensitivity of the result.
Table A.3: Typical values for material and configuration parameters

<table>
<thead>
<tr>
<th>$\rho_w (Kg , m^{-3})$</th>
<th>$T_0 (K)$</th>
<th>$\mathcal{R}_w (Jg^{-1}K^{-1})$</th>
<th>$\sigma (Nm^{-1})$</th>
<th>$P_{w1} (Pa)$</th>
<th>$R_1(\mu m)$</th>
<th>$V_{a1}/V_{fl}$</th>
<th>$c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>997</td>
<td>20 + 273</td>
<td>0.4615</td>
<td>72*10^{-3}</td>
<td>101325</td>
<td>500</td>
<td>20</td>
<td>0.85</td>
</tr>
</tbody>
</table>

The results of this investigation for the cylinder case are shown in Figure A.2.

Figure A.2: Results of evaporation analysis problem for typical values of parameters
In our second analysis we consider the effect of a change in temperature of the chamber upon the radius of the drop. This analysis problem is very similar to the previous one with the major difference that the initial and the final states have different temperatures.

State 2 considered above is chosen as the initial state and the final state is denoted state 3. Table x summarizes the equations used in this analysis.

Table A.5: Formulation of change in temperature analysis problem

<table>
<thead>
<tr>
<th></th>
<th>State 2</th>
<th>State 3</th>
</tr>
</thead>
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<tr>
<td><strong>Surface Tension</strong></td>
<td>$P_{i2} - (P_{a2} + P_{v2}) = f(R_{2})$</td>
<td>$P_{i3} - (P_{a3} + P_{v3}) = f(R_{3})$</td>
</tr>
<tr>
<td><strong>Volume of drop</strong></td>
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</tr>
<tr>
<td><strong>Ideal gas assumption for air</strong></td>
<td>$P_{a2}V_{g2} / T_{0} = P_{a3}V_{g3} / T_{3}$</td>
<td></td>
</tr>
<tr>
<td><strong>Constant total volume</strong></td>
<td>$V_{g2} + V_{i2} = V_{g3} + V_{i3}$</td>
<td></td>
</tr>
<tr>
<td><strong>Constant total mass of liquid</strong></td>
<td>$\rho_{v}(V_{i2} - V_{i3}) = \frac{1}{R_{v}} \left( \frac{P_{v3}V_{g3}}{T_{3}} - \frac{P_{v2}V_{g2}}{T_{0}} \right)$</td>
<td></td>
</tr>
<tr>
<td><strong>Equilibrium Condition</strong></td>
<td>$P_{v2} = cP_{i2}$ &amp; $P_{v3} = cP_{i3}$</td>
<td></td>
</tr>
</tbody>
</table>

Combining the equations together and simplifying, the functional form of the analysis problem is:

\[
\left[ \frac{1}{c} - 1 \right] \rho_{v} R_{v} T_{0} - \frac{T_{0}}{T_{3}} f(R_{3}) \right] \left[ g(R_{2}) - g(R_{3}) \right] - \left[ P_{a2} + \frac{T_{0}}{T_{3}} f(R_{2}) - \left( \frac{1}{c} - 1 \right) P_{v2} \right] V_{g2} = 0
\]

Or

In our second analysis we consider the effect of a change in temperature of the chamber upon the radius of the drop. This analysis problem is very similar to the previous one with the major difference that the initial and the final states have different temperatures.

State 2 considered above is chosen as the initial state and the final state is denoted state 3. Table x summarizes the equations used in this analysis.

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\[
\left[ \frac{1}{c} - 1 \right] \rho_{v} R_{v} T_{0} - \frac{T_{0}}{T_{3}} f(R_{3}) \right] \left[ g(R_{2}) - g(R_{3}) \right] - \left[ P_{a2} + \frac{T_{0}}{T_{3}} f(R_{2}) - \left( \frac{1}{c} - 1 \right) P_{v2} \right] V_{g2} = 0
\]

Or
\[
\left[ \frac{1}{c} + 1 \right] \rho \mathcal{R} T_0 - \frac{T_0}{T_3} f(R_3) \left(g(R_1) - g(R_3)\right) - \left(P_{\text{at}} + \frac{T_0}{T_3} f(R_3)\right)V_{g1} = 0
\]

Figure A.3 shows the result for the change in temperature from $15^\circ C$ to $50^\circ C$.

From Figures A.2 and A.3, it may be observed that the sensitivity of the final radius of the drop to the configuration parameters is relatively low and with the reasonable perturbation of these parameters, maximum of 10% change in final radius of the bridge will occur which will only contribute to a negligible change in force level.
Appendix B: Electrical Time Constant

An analysis of the charging of the device enables the determination of an electrical time constant with each element treated as lumped impedance, see Figure B.1, the transfer function between the potential applied to the device, $v$, and that across the dielectric layer is:

$$\frac{v_d(s)}{v(s)} = \frac{Z(s)}{2 Z(s) + R_l}$$  \hspace{1cm} (B.1)

where $Z(s)$ is the parallel impedance for the dielectric:

$$Z(s) = \frac{1}{\frac{1}{R_d} + C_d s}$$  \hspace{1cm} (B.2)

where $R_d$ and $C_d$ are the resistance and the capacitance of the dielectric layer, respectively, and $R_l$ denotes the resistance of the liquid bridge. Simplifying the equation yields:

$$\frac{v_d(s)}{v(s)} = \frac{R_d}{2R_d + R_l + R_l R_d C_d s}$$  \hspace{1cm} (B.3)

Figure B.1: Electrical model used for analysis of time constant
In this model the electrical dynamic of the device constitutes a first order system with a pole given by:

\[ s_e = -\frac{1}{C_d} \left( \frac{2}{R_i} + \frac{1}{R_d} \right) \]  

(B.4)

The resistance and the capacitance of the dielectric layer may be written in terms of the layer’s properties:

\[ R_d = \rho_d \frac{t_d}{A} \]  

(B.5)

\[ C_d = \varepsilon_d \frac{A}{t_d} \]  

(B.6)

where the parameters \( A \) and \( t \) denote the wetted area and the thickness of the dielectric layer and the permittivity and the electrical resistivity of the layer are denoted by \( \varepsilon_d \) and \( \rho_d \), respectively. The resistance of the bridge may be expressed as:

\[ R_i = \rho_l \frac{h}{A} \]  

(B.7)

where \( \rho_l \) denotes the electrical resistivity of the liquid. Substitution yields:

\[ s_e = -\frac{1}{\varepsilon_d} \left( \frac{t_d}{h} \frac{2}{\rho_l} + \frac{1}{\rho_d} \right) \]  

(B.8)

The time constant is therefore:

\[ \tau_e = \frac{\varepsilon_d}{\frac{t_d}{h} \frac{2}{\rho_l} + \frac{1}{\rho_d}} \]  

(B.9)

For most devices \( R_d >> R_i \), hence \( \rho_d t_d >> \rho_l h \). Therefore, a useful approximation of the time constant may be found:
\[
\tau_c \approx \frac{1}{2} \varepsilon_d \rho \frac{h}{t_d}
\]  
(B.10)

Substitution of some typical values for the parameters

\[
\varepsilon_d = \varepsilon_d \varepsilon_r = (8.85 \times 10^{-12}) \times 10
\]
\[
t_d = 0.1 \mu m
\]
\[
h = 30 \mu m
\]
\[
\rho_j = 18 \times 10^4 \ \Omega m \ (water)
\]
\[
\rho_d = 5 \times 10^{10} \ \Omega m \ (silicon)
\]

yields a representative value of the time constant

\[
\tau_c = 2.39 \text{ms}
\]
Appendix C: Solution of Laplace Differential Equation
for the Capillary Bridges

Starting with the Equation 2.48 in Kralchevsky [2], we have:

\[
\frac{dz}{dr} = \pm \frac{1}{|\kappa|} \frac{\kappa(r^2 - r_0^2) + r_0}{\sqrt{(r^2 - r_0^2)(r_1^2 - r^2)}}, \quad r_1 = \frac{1 - \kappa r_0}{\kappa} \quad (C.1)
\]

This can be expressed in the form:

\[
\frac{dz}{dr} = \pm \left[ \text{Sign}(\kappa) \frac{r^2 - r_0^2}{r_1^2 - r^2 + r_0} \frac{1}{|\kappa| \sqrt{(r_1^2 - r^2)(r^2 - r_0^2)}} \right] \quad (C.2)
\]

The signs “+” and “−” stand for the two branches of the generatrix that are situated above and below the plane of symmetry, \( z = 0 \).

Applying the boundary condition \( z(r_0) = 0 \), two different cases will be considered:

1. Capillary bridge with “neck”: \( r_0 \leq r < r_1 \)

Integrating the aforementioned equation, we obtain:

\[
z(r) = \pm \left[ \text{Sign}(\kappa) \int_{r_0}^{r} \frac{r^2 - r_0^2}{r_1^2 - r^2} dr + \int_{r_0}^{r} \frac{1}{|\kappa| \sqrt{(r_1^2 - r^2)(r^2 - r_0^2)}} dr \right] \quad (C.3)
\]

In the tables of integrals by Gradshtein & Ryzhik [3, 4] we find the expressions:

Eq. 3.169.11: \( \int_{\mu}^{u} \frac{\eta^2 - x^2}{\eta^2 - x^2} dx = \eta E(\phi, q) - \frac{\mu^2}{\eta} F(\phi, q) - \frac{1}{u} \sqrt{(\eta^2 - u^2)(u^2 - \mu^2)} \) \quad (C.4)

Eq. 3.152.9: \( \int_{\mu}^{u} \frac{dx}{\eta \sqrt{(\eta^2 - x^2)(x^2 - \mu^2)}} = \frac{1}{\eta} F(\phi, q) \) \quad (C.5)
\[ \eta > u > \mu > 0; \quad \sin \phi = \frac{\eta}{u} \sqrt{\frac{u^2 - \mu^2}{\eta^2 - \mu^2}}; \quad q = \frac{\sqrt{\eta^2 - \mu^2}}{\eta} \]

Substituting \( u = r \), \( \eta = r_1 \), and \( \mu = r_0 \) in the equations above, we may simplify this equation to yield:

\[
\begin{align*}
z(r) &= \pm \left\{ \text{Sign}(\kappa) \left[ r_1 E(\phi, q) - \frac{r_0^2}{r_1} F(\phi, q) - \frac{1}{r} \sqrt{(r_1^2 - r^2)(r^2 - r_0^2)} \right] + \frac{r_0}{|\kappa|} \left[ \frac{1}{r_1} F(\phi, q) \right] \right\}
\end{align*}
\]

Multiplying the equation above by \( \text{Sign}(\kappa) \) and simplifying, we find:

\[
\begin{align*}
z(r) &= \pm \left\{ r_1 E(\phi, q) + \left[ \frac{r_0}{r_1} (-r_0 + \frac{1}{\kappa}) \right] F(\phi, q) - \frac{1}{r} \sqrt{(r_1^2 - r^2)(r^2 - r_0^2)} \right\}
\end{align*}
\]

Nondimensional version of the equation above would be:

\[
\begin{align*}
z(r) &= \pm \left\{ \frac{1}{|\kappa|} \left[ r_1 |\kappa| E(\phi, q) + \left[ \frac{r_0 |\kappa|}{r_1 |\kappa|} (-r_0 |\kappa| + |\kappa|) \right] F(\phi, q) - \frac{1}{r |\kappa|} \sqrt{(r_1^2 |\kappa|^2 - r^2 |\kappa|^2)(r^2 |\kappa|^2 - r_0^2 |\kappa|^2)} \right\}
\end{align*}
\]

Simplification yields:

\[
\begin{align*}
z(r) &= \pm \left\{ \frac{1}{|\kappa|} \left[ \rho_1 E(\phi, q) + \left[ \frac{\rho_0}{\rho_1} (-\rho_0 + \text{Sign}(\rho)) \right] F(\phi, q) - \frac{1}{\rho} \sqrt{(\rho_1^2 - \rho^2)(\rho^2 - \rho_0^2)} \right\}
\end{align*}
\]

Here, we have used the fact that:

\[
\rho = |\kappa| r_1, \quad \rho_0 = |\kappa| r_1, \quad \rho = |\kappa| r
\]

The coefficient of function \( F \) in Eq. (C.9) can be further simplified to achieve:
\[ C = \frac{P_0}{\rho_1} \left( -\rho_0 + \text{Sign}(p) \right) = \frac{\text{Abs}[p]}{\text{Abs}[1 - p]} \left( -\text{Abs}[p] + \text{Sign}(p) \right) \]

\[ P < 1/2 \Rightarrow \text{Abs}[1 - p] = 1 - p \]

\[ \Rightarrow C = \frac{\text{Abs}[p]}{1 - p} \left( -\text{Abs}[p] + \text{Abs}[p]/p \right) = \frac{\text{Abs}[p]^2}{1 - p} \left( -1 + \frac{1}{p} \right) = \frac{p^2}{1 - p} \left( -p + 1 \right) = p \]

The final result is then:

\[ z(r) = \pm \frac{1}{|\kappa|} \left\{ \rho_1 E(\phi, q) + \text{Sign}(p) \rho_0 F(\phi, q) - \frac{1}{p} \sqrt{(\rho_1^2 - \rho^2)(\rho^2 - \rho_0^2)} \right\} \]

(C.12)

2. Meniscus with “haunch”: \( r_1 < r \leq r_0 \)

Integrating Eqn. (C.1), we obtain:

\[ z(r) = \pm \left[ -\text{Sign}(\kappa) \int_{r_1}^{r} \sqrt{r_0^2 - r'^2} \, dr' + \int_{r}^{r_0} \frac{dr'}{\sqrt{(r_0^2 - r'^2)(r^2 - r_1^2)}} \right] \]

(C.13)

In the tables of integrals by Gradshtein & Ryzhik [3, 4] we find the expressions:

\[ \text{Eq. 3.169.17:} \int_{u}^{\eta} \frac{\eta^2 - x^2}{\sqrt{x^2 - \mu^2}} \, dx = \eta [F(\phi, q) - E(\phi, q)] \]

(C.14)

\[ \text{Eq. 3.152.10:} \int_{u}^{\eta} \frac{dx}{\sqrt{(\eta^2 - x^2)(x^2 - \mu^2)}} = \frac{1}{\eta} F(\phi, q) \]

(C.15)

\[ \eta > u > \mu > 0; \quad \sin \lambda = \frac{\eta^2 - u^2}{\sqrt{\eta^2 - \mu^2}}; \quad q = \frac{\sqrt{\eta^2 - \mu^2}}{\eta} \]
Substituting $u = r$, $\eta = r_0$, and $\mu = r_i$ in the equations above, we may simplify the equation to find:

$$z(r) = \pm \left\{ -\text{Sign}(\kappa) r_0 [F(\phi, q) - E(\phi, q)] + \frac{r_0}{|\kappa|} F(\phi, q) \right\} \quad \text{(C.16)}$$

$$z(r) = \pm \left\{ r_0 E(\phi, q) + \left[ \frac{1}{\kappa} - r_0 \right] F(\phi, q) \right\} \quad \text{(C.17)}$$

Nondimensional version of the equation above is:

$$z(r) = \pm \frac{1}{|\kappa|} \left\{ r_0 |\kappa| E(\phi, q) + \left[ \frac{|\kappa|}{\kappa} - r_0 |\kappa| \right] F(\phi, q) \right\} \quad \text{(C.18)}$$

Simplification yields:

$$z(r) = \pm \frac{1}{|\kappa|} \left\{ \rho_0 E(\phi, q) + [1 - \rho_0] F(\phi, q) \right\} \quad \text{(C.19)}$$

In order to resolve any ambiguity in the calculations, the definition of the elliptic integrals used in the equations must be included.

$$F(\phi, q) = \int_0^\phi \frac{d\nu}{\sqrt{1 - q^2 \sin^2(\nu)}} = \int_0^{\sin \phi} \frac{dt}{\sqrt{(1 - t^2)(1 - q^2 t^2)}}$$

$$E(\phi, q) = \int_0^\phi \sqrt{1 - q^2 \sin^2(\nu)} d\nu = \int_0^{\sin \phi} \sqrt{\frac{1 - q^2 t^2}{1 - t^2}} dt$$
Appendix D: Proof of Unduloid Profiles Symmetry around Cylinder Case

In this appendix we will use the analytical equations available for unduloid profiles to prove that profile shape of unduloids with dimensionless pressures $p$ and $1-p$ are equivalent under a transformation of axial coordinate. To do so, we consider two unduloids with dimensionless pressures $\hat{p}$ and $1-\hat{p}$ as shown in Figure D.1. The first is an unduloid with neck, the neck having a nondimensional radius of $\rho_0 = \hat{p}$. The later is an unduloid with haunch, the radius of which is $\rho_0 = 1 - \hat{p}$.

![Figure D.1: a) Unduloid with dimensionless pressure $\hat{p}$. b) Unduloid with dimensionless pressure $1-\hat{p}$.](image)

We will consider sections of these two unduloids, each defining a capillary bridge, chosen so as to have the same arbitrary constant radii, $\rho_c$. We will show that the sum of
the dimensionless height of these two bridges for any prescribed constant radius is constant and is equal to the height of either bridge from neck to haunch. This distance will depend only upon the dimensionless pressure and is independent of the prescribed constant radius. That is:

$$\zeta(\rho_c)\big|_{p=p} + \zeta(\rho_c)\big|_{p=1-p} = q(\hat{p})$$

To prove this we will start with the analytical expressions available for the capillary bridge and substitute in the values of $\rho_0$ and $\rho_1$ for each bridge in terms of the dimensionless pressure for the bridge. The explicit expressions for the elliptical integrals are also used. This approach yields an expression for the height of the first bridge:

$$\zeta(\rho_c)\big|_{p=p} = \left[ \rho_c E\left(\sin^{-1}\sqrt{1-\rho_0^2 / \rho_c^2}, \sqrt{1-\rho_0^2 / \rho_c^2}\right) + \rho_0 F\left(\sqrt{1-\rho_0^2 / \rho_c^2}, \sqrt{1-\rho_0^2 / \rho_c^2}\right) \right]_{\rho_0=\hat{p}, \rho_1=1-\hat{p}}$$

$$= (1-\hat{p}) E\left(\sin^{-1}\sqrt{1-\hat{p}^2 / \rho_c^2}, \sqrt{1-\hat{p}^2 / (1-\hat{p})^2}\right)$$

$$+ \hat{p} F\left(-\sqrt{1-\hat{p}^2 / \rho_c^2}, \sqrt{1-\hat{p}^2 / (1-\hat{p})^2}\right) - \sqrt{(\rho_c^2 - \hat{p}^2)(1-\hat{p}^2 - \rho_c^2) / \rho_c} \int_0^{\sqrt{1-\hat{p}^2 / (1-\hat{p})^2}} \frac{u^2}{\sqrt{1-(1-\hat{p}^2 / (1-\hat{p})^2)u^2}} du$$

$$+ \hat{p} \int_0^{\sqrt{1-\hat{p}^2 / (1-\hat{p})^2}} \frac{du}{\sqrt{(1-u^2)(1-(1-\hat{p}^2 / (1-\hat{p})^2)u^2)}}$$

And for the second bridge:
\[ \zeta(\rho_c) |_{p=1-p} = [\rho_0E(\sin^{-1} \frac{\sqrt{1-\rho_c^2/\rho_0^2}}{\sqrt{1-\rho_1^2/\rho_0^2}}, \sqrt{1-\rho_1^2/\rho_0^2}) + (1-\rho_0)F(\sqrt{1-\rho_c^2/\rho_0^2}, \sqrt{1-\rho_1^2/\rho_0^2})]_{\rho_0=1-p, \rho_1=p} \]

\[ = (1-\hat{\rho})E(\sin^{-1} \frac{\sqrt{1-\rho_c^2/(1-\hat{\rho})^2}}{\sqrt{1-\hat{\rho}^2/(1-\hat{\rho})^2}}, \sqrt{1-\hat{\rho}^2/(1-\hat{\rho})^2}) \]

\[ + \hat{\rho} F(\sqrt{1-\rho_c^2/(1-\hat{\rho})^2}, \sqrt{1-\hat{\rho}^2/(1-\hat{\rho})^2}) \]

\[ = (1-\hat{\rho}) \int_{0}^{\hat{\rho}} \left[ \frac{1-(1-\hat{\rho}^2/(1-\hat{\rho})^2)u^2}{1-u^2} \right] du \]

\[ + \hat{\rho} \int_{0}^{\hat{\rho}} \frac{du}{\sqrt{1-u^2}(1-(1-\hat{\rho}^2/(1-\hat{\rho})^2)u^2)} \]

The sum of the height of these two bridges is:

\[ \zeta(\rho_c) |_{p=\hat{\rho}} + \zeta(\rho_c) |_{p=1-\hat{\rho}} = \int_{0}^{\hat{\rho}} \left[ (1-\hat{\rho}) \sqrt{1-(1-\hat{\rho}^2/(1-\hat{\rho})^2)u^2} \right. \]

\[ + \frac{\hat{\rho}}{\sqrt{(1-u^2)(1-(1-\hat{\rho}^2/(1-\hat{\rho})^2)u^2)}} \left. \right] du \]

\[ - \sqrt{(\rho_c^2-\hat{\rho}^2)((1-\hat{\rho}^2-\rho_c^2)) / \rho_c} \]

\[ + \int_{0}^{\hat{\rho}} \left[ (1-\hat{\rho}) \sqrt{1-(1-\hat{\rho}^2/(1-\hat{\rho})^2)u^2} \right. \]

\[ + \frac{\hat{\rho}}{\sqrt{(1-u^2)(1-(1-\hat{\rho}^2/(1-\hat{\rho})^2)u^2)}} \left. \right] du \]
To show the independence of this expression from the prescribed constant radius $\rho_c$, we will take the first derivative with respect to $\rho_c$ and simplify, employing the generalized form of fundamental theorem of calculus:

$$
\frac{d}{d\rho_c}[\xi(\rho_c)|_{p=\hat{p}} + \xi(\rho_c)|_{p=1-\hat{p}}] = \frac{d}{d\rho_c} \left[ \frac{\sqrt{1 - \hat{p}^2 / \rho^2}}{\sqrt{1 - \hat{p}^2 / (1 - \hat{p})^2}} \right] \times \left[ (1 - \hat{p}) \sqrt{\frac{1 - (1 - \hat{p}^2) / (1 - \hat{p})^2}{1 - u^2}} \right] 
$$

$$
+ \frac{\hat{p}}{\sqrt{(1 - u^2)(1 - (1 - \hat{p}^2 / (1 - \hat{p})^2)) u^2}} \frac{\sqrt{\rho^2}}{\sqrt{(1 - u^2)(1 - (1 - \hat{p}^2 / (1 - \hat{p})^2)) u^2}}
$$

$$
- \frac{d}{d\rho_c} \left[ \sqrt{(\rho^2 - \hat{p}^2)((1 - \hat{p})^2 - \rho^2)} / \rho \right] 
$$

$$
+ \frac{d}{d\rho_c} \left[ \frac{\sqrt{1 - \rho^2 / (1 - \hat{p})^2}}{\sqrt{1 - \hat{p}^2 / (1 - \hat{p})^2}} \right] \times \left[ (1 - \hat{p}) \sqrt{\frac{1 - (1 - \hat{p}^2) / (1 - \hat{p})^2}{1 - u^2}} \right] 
$$

$$
+ \frac{\hat{p}}{\sqrt{(1 - u^2)(1 - (1 - \hat{p}^2 / (1 - \hat{p})^2)) u^2}} \frac{\sqrt{\rho^2}}{\sqrt{(1 - u^2)(1 - (1 - \hat{p}^2 / (1 - \hat{p})^2)) u^2}}
$$

Simplifying yields:

$$
\frac{d}{d\rho_c}[\xi(\rho_c)|_{p=\hat{p}} + \xi(\rho_c)|_{p=1-\hat{p}}] = \frac{1}{\sqrt{1 - \hat{p}^2 / (1 - \hat{p})^2}} \left[ \frac{\hat{p}^2}{\rho^2} \right] 
$$

$$
\times \left[ (1 - \hat{p}) \sqrt{\frac{\hat{p}^2 / \rho^2}{1 - \left( \frac{1 - \hat{p}^2 / \rho^2}{1 - \hat{p}^2 / (1 - \hat{p})^2} \right)}} \right] 
$$

$$
+ \frac{\hat{p}}{\sqrt{(1 - \left( \frac{1 - \hat{p}^2 / \rho^2}{1 - \hat{p}^2 / (1 - \hat{p})^2} \right))(\hat{p}^2 / \rho^2)}} 
$$

$$
+ \frac{\sqrt{(1 - \hat{p})^2 - \rho^2}}{\rho^2} \left( \frac{\rho^2 - \hat{p}^2}{\sqrt{(1 - \hat{p})^2 - \rho^2}(\rho^2 - \hat{p}^2)} \right) \frac{(1 - \hat{p})^2 + \hat{p}^2 - 2 \rho^2}{\sqrt{(1 - \hat{p})^2 - \rho^2}(\rho^2 - \hat{p}^2)}
$$
This confirms \( \zeta(\rho_c)\big|_{p=\hat{p}} + \zeta(\rho_c)\big|_{p=1-\hat{p}} \) is independent of the prescribed constant radius \( \rho \).

Hence, we write:

\[
\zeta(\rho_c)\big|_{p=\hat{p}} + \zeta(\rho_c)\big|_{p=1-\hat{p}} = q(\hat{p})
\]

Substituting \( \rho_c = \hat{p} \) in the equation above yields:

\[
\zeta(\rho_c)\big|_{p=\hat{p}} + \zeta(\rho_c)\big|_{p=1-\hat{p}} = \zeta(\hat{p})\big|_{p=\hat{p}} + \zeta(\hat{p})\big|_{p=1-\hat{p}} = q(\hat{p})
\]

And substitution of \( \rho_c = 1 - \hat{p} \) yields:

\[
\zeta(\rho_c)\big|_{p=\hat{p}} + \zeta(\rho_c)\big|_{p=1-\hat{p}} = \zeta(1 - \hat{p})\big|_{p=\hat{p}} + \zeta(1 - \hat{p})\big|_{p=1-\hat{p}} = q(\hat{p})
\]

But we know that

\[
\zeta(\hat{p})\big|_{p=\hat{p}} = \zeta(1 - \hat{p})\big|_{p=1-\hat{p}} = 0
\]

Therefore
\[ q(\hat{p}) = \zeta(\hat{p}) \big|_{p=1-\rho} = \zeta(1 - \hat{p}) \big|_{p=\rho} \]

which is the height of each bridge from neck to haunch which we denote \( \overline{\zeta}(\hat{p}) \). Hence, we have for any \( \rho_c \)

\[ \zeta(\rho_c) \big|_{p=\rho} + \zeta(\rho_c) \big|_{p=1-\rho} = \overline{\zeta}(\hat{p}) \]
Appendix E: Analytical Approximation of Geometrical Equations for Low Aspect Ratio Bridges

In order to derive the approximate equation for the geometrical parameters of a low aspect ratio capillary bridge, we will consider it as a surface of revolution of a circular arc. Two possible scenarios are neck and haunch, illustrated in Figure E.1.

As it is shown in the figures, \( r_\perp \) denotes the radius of the arc and \( h \) denotes the height of the bridge. Using this notation, the geometrical relations in Table E.1 may be derived. The equations for volume and liquid/gas surface area \( S_{gl} \) are derived using assumptions that \( \theta \) is close to \( \pi/2 \) (i.e., \( r_0 \approx r_\perp \)) and that \( r \gg h \), in which \( r \) may be used interchangeably with \( r_0 \) and \( r_\perp \). Ignoring the higher order terms in the equations for the surface \( S_{gl} \) and the volume of the bridge yields:

\[
S_{gl} = 2\pi rh \quad (E.1)
\]

\[
V = \pi r^2 h \quad (E.2)
\]
Of course, we always find that:

$$S_{ls} = \pi r^2$$  \hspace{1cm} (E.3)

Table E.1: Geometrical relations for surface of a revolution of an arc

<table>
<thead>
<tr>
<th></th>
<th>$r_c - r_0$</th>
<th>$r_\perp (1 - \sin(\theta))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h$</td>
<td>$2 , r_\perp \cos(\theta)$</td>
<td></td>
</tr>
</tbody>
</table>

Bridge with neck
$(\theta < \pi/2)$

<table>
<thead>
<tr>
<th>$S_{gl}$</th>
<th>$2 \left[ r_{\perp - \theta}^2 - 2\pi (r_0 + r_\perp - r_\perp \cos(\phi)) r_\perp \cos(\phi) , d\phi \right.$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$2\pi r , h + h^2 \left( -\frac{1}{2} \pi \sec(\theta) - \frac{1}{4} \pi^2 \sec^2(\theta) + \frac{1}{2} \pi \theta \sec^2(\theta) \right)$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$V$</th>
<th>$2 \left[ r_{\perp - \theta}^2 - \pi (r_0 + r_\perp - r_\perp \cos(\phi))^2 r_\perp \cos(\phi) , d\phi \right.$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\pi r^2 , h + h^2 \left( -\frac{1}{2} \pi \sec(\theta) - \frac{1}{4} \pi^2 \sec^2(\theta) + \frac{1}{2} \pi \theta \sec^2(\theta) \right)$</td>
</tr>
</tbody>
</table>

Bridge with haunch
$(\theta > \pi/2)$

<table>
<thead>
<tr>
<th>$S_{gl}$</th>
<th>$2 \left[ r_{\perp - \theta}^2 - \pi (r_0 + r_\perp + r_\perp \cos(\phi))^2 r_\perp \cos(\phi) , d\phi \right.$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$2\pi r , h + h^2 \left( -\frac{1}{2} \pi \sec(\theta) - \frac{1}{4} \pi^2 \sec^2(\theta) + \frac{1}{2} \pi \theta \sec^2(\theta) \right)$</td>
</tr>
</tbody>
</table>

Describing $S_{gl}$ and $S_{ls}$ in terms of $V$ and $h$ we find that:

$$S_{gl} = 2\sqrt{\pi V h}$$  \hspace{1cm} (E.4)

$$S_{ls} = \frac{V}{h}$$  \hspace{1cm} (E.5)

Taking derivative of these two in terms of $h$, we will have:
\[ \frac{\partial S_{gl}}{\partial h}_{v \text{ fixed}} = \sqrt{\pi} \frac{\sqrt{V}}{h} = \pi r \] (E.6)

\[ \frac{\partial S_{sh}}{\partial h}_{v \text{ fixed}} = -\frac{V^2}{h} = -\pi \frac{r^2}{h} = -\pi \left( \frac{r}{h} \right)^2 \] (E.7)

For the case of bridge with unequal contact angles one may obtain (Fig. E.2):

\[ h = h_1 + h_2 = r_\perp \cos(\theta_i) + r_\perp \cos(\theta_u) = r_\perp (\cos(\theta_i) + \cos(\theta_u)) \] (E.8)

Assuming bridge is a pancake shape, Eqns. (E.6) and (E.7) will be still valid for the case of unequal contact angles.

Figure E.2: Approximation of capillary bridge with unequal
Appendix F: Relationship between Force Equations

As it has been discussed in Section 2.3.2, the capillary force of a cylindrical bridge may be calculated using two different equations. In this section we will show the equivalence of these two for the low aspect ratio bridges. Reintroducing two equations from Section 2.3.2 we have:

\[
\frac{F_c}{\sigma_{gl}} = -2\pi r_0 (1 - p) \quad (F.1)
\]

\[
\frac{F_c}{\sigma_{gl}} = -\frac{\partial S_{gl}}{\partial h} + 2\cos(\theta_0) \frac{\partial S_{ls}}{\partial h} \quad (F.2)
\]

To prove the equivalence of these two we start from Eqn. (F.2) and do substitutions to arrive at Eqn. (F.1). From Appendix C, we know that for a drop with low aspect ratio we have:

\[
\frac{\partial S_{gl}}{\partial x} \bigg|_{r \text{ fixed}} = \pi r \quad (F.3)
\]

\[
\frac{\partial S_{ls}}{\partial x} \bigg|_{r \text{ fixed}} = -\pi \frac{r^2}{h} \quad (F.4)
\]

Here it has been assumed that \( r_c \approx r_0 \) and the notation \( r \) has been used for all the radii.

Substitution of these equations into Eqn. (F.2) yields:

\[
\frac{F_c}{\sigma_{gl}} = -\pi r + 2\cos(\theta_0)\left(-\frac{\pi r^2}{h}\right) \quad (F.5)
\]

Or,

\[
\frac{F_c}{\sigma_{gl}} = -2\pi r \left[1 - \left(\frac{1}{2} - \cos(\theta_0)\left(\frac{r}{h}\right)\right)\right] \quad (F.6)
\]
From Table E.1 in Appendix E we have that:

\[\cos \frac{\theta}{h} = \pm \frac{1}{2r_\perp}\]  \hspace{1cm} (F.7)

where \(r_2\) is the radius of the circular arc and the sign depends on whether the bridge has a neck or haunch. Substitution of Eqn. (F.7) into Eqn. (F.6) and rearranging yields:

\[\frac{F_c}{\sigma_{gl}} = -2\pi r \left[1 - \left(\frac{r}{2}\right)\left(\frac{1}{r} \pm \frac{1}{r_\perp}\right)\right]\]  \hspace{1cm} (F.8)

On the other side Laplace equation states that:

\[P_c = \sigma \left[\frac{1}{r} \pm \frac{1}{r_\perp}\right]\]  \hspace{1cm} (F.9)

In which the sign depends on whether the bridge has a neck or haunch and pressure difference over interface \(P_c\) and dimensionless pressure \(p\) are related as:

\[P_c = \frac{2\sigma}{r} p\]  \hspace{1cm} (F.10)

Combination of (F.9) and (F.10) yields:

\[p = \left(\frac{r}{2}\right)\left[\frac{1}{r} \pm \frac{1}{r_\perp}\right]\]  \hspace{1cm} (F.11)

Substituting (F.11) into (F.8) we get:

\[\frac{F_c}{\sigma_{gl}} = -2\pi r_0 \left(1 - p\right)\]  \hspace{1cm} (F.12)

which is the same as (F.1) and therefore we can conclude that the equations (F.1) and (F.2) are equivalent.
Appendix G: Electromechanical Analysis of Bridge

Profile near the Contact Line

Determining the bridge profile close to contact line requires the distribution of electric field around the contact line which itself depends on the local bridge shape. Here, an iterative procedure similar to that of Mugele et al. [5] is employed, in which the electric field and bridge shape are determined successively, each with fixing the other one. To do this numerically, an electrostatic boundary condition problem as shown in Figure G.1 is considered.

Figure G.1: Geometry and boundary conditions of electromechanical analysis

The liquid is assumed to be perfectly conductive and thus electric field is zero inside the bridge ($\vec{E} = 0$) and normal to the bridge profile on the gas-liquid interface ($\vec{E} \cdot \vec{t} = 0$ where $\vec{t}$ is the local tangent vector). The bridge profile is represented as a function $x = f(y)$, initially set to be a line ($P_c = 0$) with a tangent angle equal to the apparent contact angle ($\theta_v$). In each iteration step, first the electrostatic problem is solved...
using the finite element package FEMM [6]. Then starting from the fixed point A the profile shape is found by integrating the surface curvature resulted from electrostatic stress on the surface:

$$\frac{f''}{(1+f'^2)^{3/2}} = k = \frac{\varepsilon_0}{2\sigma_{gl}} \tilde{E}(x,y)^2$$  \hspace{1cm} (G.1)

While integrating Equation (E.1) the electrostatic field is always captured at the points on the drop profile in the former iteration step. In other words, once the new profile in terms of vertical coordinate axis \((x = f'_{new}(y))\) deviates from the former profile \((x = f'_{old}(y))\), the electrostatic field at the point \((x, f'_{old}(y))\) is used rather than point \((x, f'_{new}(y))\). The later might potentially fall inside or outside the former bridge profile resulting in significantly less electrostatic field intensity.

In order to capture the intense electric field and the correct behavior of bridge profile very close to contact line, very fine meshing of the area in required. For grid size in order of nanometers, the local contact angle is found to merge within ten iterations and matches the value predicted by Young-Lippmann equation:

$$\cos(\theta_0) = \frac{\varepsilon_d}{2\sigma_{gl}I_d}v_d^2 + \cos(\theta_v)$$  \hspace{1cm} (G.2)

In addition, one may compute the force components \(\tilde{F}^r_E\) and \(\tilde{F}^z_E\) introduced in Section 2.4.1 by integrating the horizontal component of electrostatic field on the drop profile and vertical component of the field on the solid/gas interface, respectively. As shown in Section 2.4.1, these two forces are expected to be:

$$\tilde{F}^r_E = \frac{\varepsilon_d}{2t_d}v_d^2$$  \hspace{1cm} (G.3)
and

\[ \tilde{F}^z_{\text{gl}} = \sigma_{\text{gl}} \left[ \sin \theta_0 - \sin \theta_v \right] \quad (G.4) \]

As a numerical evidence for the theory proposed and explained in Section 2.4.1, we consider a system with apparent contact angle of 60°. The dielectric constant and thickness are assumed to be 1.8 and 1 \( \mu m \), respectively. A voltage of 67.6 volts is applied to result in change of cosine equal to:

\[ n = \frac{1}{2 \sigma_{\text{gl}}} \varepsilon_0 \varepsilon_r v_d^2 \approx 0.5 \]

The native contact angle was expected to be 90° and was numerically observed to be 90.5°. Radial component of electrostatic force on the bridge profile (\( \tilde{F}^r_{\text{gl}} \)) and axial component of electrostatic force on the solid/gas surface (\( \tilde{F}^z_{\text{gl}} \)) were computed to be 0.49 and 0.064, respectively. These numbers are within 3% error of the expected values of \( \frac{\varepsilon_d}{2t_d} v_d^2 = 0.5 \), \( \sigma_{\text{gl}} \left[ \sin \theta_0 - \sin \theta_v \right] = 0.062 \), respectively.
Appendix H: Numerical Approach

To speed the numerical solution to the axisymmetric capillary bridge problem and also be able to add the electric field effects in the solution method, an algorithm similar to that of Surface Evolver may be implemented. The profile shape may be discretized and approximated by a piecewise linear curve. To do so a number of nodes \( m \) numbered from bottom to top with two coordinates \( r_{(i)}, z_{(i)} \) (where \( i \) is the number of node) is considered and the profile is formed with straight lines between successive nodes (see Fig. H.1). The vector containing all the components of the nodes is called coordinates vector and is denoted as \( \vec{c} \).

![Figure H.1: Approximation of capillary bridge profile as a piecewise linear curve](image)

Similar to Surface Evolver, an iterative method is used to minimize the total energy of the system. This is done by finding the gradient of the energy with respect to a change in coordinates of the nodes defining the bridge profile and moving the nodes appropriately to minimize the total energy. The total energy of the system is calculated using:

\[
E = \sigma_{gl}(S_{gl} - 2S_{ls} \cos(\theta_0)) - \frac{1}{4} \varepsilon_d S_{ls} \frac{S_{ls}}{l_d} R_{l} \left( \frac{1}{1 + \frac{R_l}{R_d}} \right)^2
\]  

(H.1)
where $S_{gl}$ and $S_{ls}$ denote the gas/liquid and liquid/solid interface areas and are given by:

$$S_{gl} = \sum \pi (r_{(i+1)}^2 - r_{(i)}^2) \sqrt{1 + \left(\frac{z_{(i+1)} - z_{(i)}}{r_{(i+1)} - r_{(i)}}\right)^2}$$  \hspace{1cm} (H.2)$$

$$S_{ls} = \pi r_{(0)}^2 = \pi r_{(m)}^2$$  \hspace{1cm} (H.3)$$

$R_l$ and $R_d$ denote the electrical resistance of liquid and dielectric may be calculated via the finite element routine FEMM [6]. To do so, the profile data must be translated into FEMM input. The electrical properties of the liquid and dielectric layer on top of the electrodes as well as boundary conditions will be set and a meshing will be used to determine the distribution of electric field everywhere within the boundaries. Once the electrostatic finite element problem is solved, the equivalent resistance of the liquid and dielectric may be computed from available data. Because of relatively uniform field inside the dielectric layer, the resistance of the dielectric layer may also be calculated as:

$$R_d = \rho_d \frac{l_d}{S_{ls}}$$  \hspace{1cm} (H.4)$$

These resistances are only considered to study the saturation mechanism proposed by Shapiro. One may alternatively consider the liquid as a perfect conductor in which case Eqn. (D.1) will simplify to:

$$E = \sigma_{gl} (S_{gl} - 2S_{ls} \cos(\theta_0)) - \frac{1}{4} \varepsilon_d \frac{S_{ls}}{l_d} v^2$$  \hspace{1cm} (H.5)$$

In this case it is no longer necessary to call the finite element routine. It is important to note that here, the liquid profile is crudely approximated and we do not expect to observe the conservation of local contact angle.

Volume of the bridge in each iteration may be computed as:
\[ V = \sum \frac{\pi}{3} \left( r_{(i+1)}^3 - r_{(i)}^3 \right) \left( \frac{z_{(i+1)} - z_{(i)}}{r_{(i+1)} - r_{(i)}} \right) \]  

(H.6)

Energy and volume gradient vectors \((dE, dV)\) components \((dE_{(i)}\) and \(dV_{(i)}\)) may be computed for each coordinate \(c_{(i)}\) by:

\[
dE_{(i)} = \left| E(c_{(i)} = c_{(i)} + dc) - E(c_{(i)} = c_{(i)} - dc) \right| / (2dc)
\]

(H.7)

\[
dV_{(i)} = \left| V(c_{(i)} = c_{(i)} + dc) - V(c_{(i)} = c_{(i)} - dc) \right| / (2dc)
\]

(H.8)

The displacement vector (vector containing the computed change in the coordinates of all the nodes) is found as negative of energy gradient plus a term correcting the volume change from last iteration:

\[
\bar{\Delta}c^k = \frac{1}{2^n} \left[ -\frac{\bar{d}E}{|\bar{d}E|} + \frac{\bar{d}V}{|\bar{d}V|} \Delta V \right]
\]

(H.9)

The factor \(1/2^n\) is considered to control the magnitude of the displacements imposed. This factor is decreased \((n\) is increased\) when a certain number of nodes are found to be oscillating rather than moving in one direction. The iteration process is stopped once the change in node coordinates is less than the desired accuracy. Figure H.2 summarizes the algorithm described here.
Start

Initialize the values of the coordinates of the vertices

\[
\Delta V = V_0 - V
\]

\[
dE_i = \left| E(c_i = c_i + dc) - E(c_i = c_i - dc) \right|/(2 dc)
\]

\[
dV_{(i)} = \left| V(c_i = c_i + dc) - V(c_i = c_i - dc) \right|/(2 dc)
\]

\[
\tilde{\delta}c^k = \frac{1}{2^n} \left[ \frac{-\tilde{d}E}{|\tilde{d}E|} + \frac{\tilde{d}V}{|\tilde{d}V|} \right] \Delta V
\]

\[
\tilde{c} = \tilde{c} + \tilde{\delta}c^k
\]


Figure H.2: Flow chart of numerical approach to determine bridge profile
Appendix I: Approximations for Pancake Bridge as a Surface of Revolution of Circular Arc

In this section we develop relationships for a low aspect ratio (“pancake”) bridge with its shape being the surface of revolution of a circular arc, see Figure I.1.

From the geometry, we may find

\[ \tilde{x} = \sqrt{r_{\perp}^2 - (r_{\perp} \cos(\theta) - z)^2} \]  

(I.1)

\[ r(z) = r_c + r_{\perp} \sin(\theta) - \tilde{x} \]  

(I.2)

Hence

\[ r(z) = r_c + \frac{h}{2} \tan(\theta) - r_{\perp} \sqrt{1 - \cos^2(\theta) \left(1 - \frac{2z}{h}\right)^2} \]  

(I.3)

or

Figure I.1: Approximation of a capillary bridge as a surface of a revolution of an arc.
Let $\xi = \frac{z}{h}$, $d\xi = \frac{1}{h}dz$, $dz = h d\xi$. Then

$$r(\xi) = r_c + h \frac{\tan(\theta)}{2} - \frac{h}{2} \sqrt{1 - \cos^2(\theta) (1 - 2\xi)^2} \frac{\cos(\theta)}{h} \quad (I.5)$$

We may find the bridge volume as:

$$V = \int_0^h \pi r(z)^2 \, dz = \int_0^1 \pi r(\xi)^2 \, h \, d\xi \quad (I.6)$$

Define $f(\theta, \xi)$ as:

$$f(\theta, \xi) = \frac{1}{2} \frac{\tan(\theta)}{2} - \frac{1}{2} \frac{\sqrt{1 - \cos^2(\theta) (1 - 2\xi)^2}}{\cos(\theta)} \quad (I.7)$$

Then

$$r(\xi) = r_c + h f(\theta, \xi) \quad (I.8)$$

and

$$V = \int_0^1 \pi (r_c + h f(\theta, \xi))^2 h \, d\xi = \int_0^1 \pi (r_c^2 + 2 h r_c f(\theta, \xi) + h^2 f^2(\theta, \xi)) h \, d\xi \quad (I.9)$$

Define

$$F_1(\theta) = \int_0^1 f(\theta, \xi) \, d\xi \quad \text{and} \quad F_2(\theta) = \int_0^1 f^2(\theta, \xi) \, d\xi \quad (I.10)$$

Then the volume may be written as

$$V = \pi r_c^2 h + 2 \pi r_c h^2 F_1(\theta) + \pi h^3 F_2(\theta) \quad (I.11)$$

Taking a total derivative yields:
\[
dV = 0 = [2\pi r_c h + 2\pi h^2 F_1]dr_c + [\pi r_c^2 + 4\pi r_c h F_1 + 3\pi h^2 F_2]dh
\] (I.12)

This provides a relationship between change in contact line radius and change in bridge height:

\[
dr_c = -\frac{r_c}{2h} \left[ 1 + 4\left(\frac{h}{r_c}\right)F_1 + 3\left(\frac{h}{r_c}\right)^2 F_2 \right] dh
\] (I.13)

Taylor series expansion in terms of the small quantity \(\left(\frac{h}{r_c}\right)\) yields:

\[
\begin{align*}
\frac{1 + 4\left(\frac{h}{r_c}\right)F_1 + 3\left(\frac{h}{r_c}\right)^2 F_2}{1 + \left(\frac{h}{r_c}\right)F_1} & \approx 1 + 3\left(\frac{h}{r_c}\right)F_1
\end{align*}
\] (I.14)

The wetted area is given by

\[
S_{bs} = \pi r_c^2
\] (I.15)

Hence

\[
\begin{align*}
dS_{bs} &= 2\pi r_c \cdot dr_c = -\frac{\pi r_c^2}{h} \left[ 1 + 4\left(\frac{h}{r_c}\right)F_1 + 3\left(\frac{h}{r_c}\right)^2 F_2 \right] dh
\end{align*}
\] (I.16)

or approximately

\[
\begin{align*}
dS_{bs} \approx -\frac{\pi r_c^2}{h} \left[ 1 + 3\left(\frac{h}{r_c}\right)F_1 \right] dh
\end{align*}
\] (I.17)
For a cylindrical bridge, \( dS_{\text{i}} = -\frac{\pi r^2}{h} \, dh \), the result without the correction term
\[
3 \left( \frac{h}{r_c} \right) F_1. \text{ Note that } F_1(\frac{\pi}{2}) = 0
\]

Now we will develop an approximation for \( F_1(\theta) \) from our definition
\[
F_1(\theta) = \int_0^1 f(\theta, \xi) \, d\xi \quad (I.18)
\]
\[
f(\theta, \xi) = \frac{1}{2} \tan(\theta) - \frac{1}{2} \frac{\sqrt{1 - \cos^2(\theta)(1 - 2\xi)^2}}{\cos(\theta)} \quad (I.19)
\]

Note that L’Hospital’s rule states that \( f \left( \frac{\pi}{2}, \xi \right) = 0 \) therefore, \( F_1(\frac{\pi}{2}) = 0 \). Integrating we obtain
\[
F_1(\theta) = \frac{1}{2} \tan(\theta) - \frac{1}{2} \int_0^1 \frac{\sqrt{1 - \cos^2(\theta)(1 - 2\xi)^2}}{\cos(\theta)} \, d\xi \quad (I.20)
\]

Let \( \eta = c[1 - 2\xi] \) where \( c = \cos(\theta) \). Then
\[
d\eta = -2c \, d\xi, \quad d\xi = -\frac{1}{2c} \, d\eta \quad (I.21)
\]
\[
\xi = 0 \Rightarrow \eta = c \text{ and } \xi = 1 \Rightarrow \eta = -c
\]

\[
-\frac{1}{2c} \int_0^c \frac{\sqrt{1 - \cos^2(\theta)(1 - 2\xi)^2}}{\cos(\theta)} \, d\xi = -\frac{1}{2c} \int_c^{-c} \frac{\sqrt{1 - \eta^2}}{\cos(\theta)} d\eta = -\frac{1}{2c} \int_c^{-c} \frac{\sqrt{1 - \eta^2}}{\cos(\theta)} d\eta
\]
\[
= -\frac{1}{2c} \int_c^{-c} \sqrt{1 - \eta^2} \, d\eta = -\left( \frac{1}{2c} \right)^2 2 \int_0^c \sqrt{1 - \eta^2} \, d\eta = -\left( \frac{1}{2c} \right)^2 \left[ c\sqrt{1 - c^2} + \sin^{-1}(c) \right] \text{ (from integral table)}
\]
\[
= -\frac{1}{4} \left[ \frac{\sin(\theta)}{\cos(\theta)} + \frac{\sin^{-1}(\cos(\theta))}{\cos^2(\theta)} \right]
\]
L’Hospital’s rule shows this is zero at \( \theta = \frac{\pi}{2} \). (One may apply the rule twice to demonstrate this result)

\[
F_1(\theta) = \frac{1}{2} \tan(\theta) - \frac{1}{4} \left[ \sin(\theta) + \sin^{-1}(\cos(\theta)) \right] = \frac{1}{4} \left[ \tan(\theta) - \frac{\pi/2 - \theta}{\cos^2(\theta)} \right]
\]

(1.23)

\[
dS_{ls} \approx -\pi r_c^2 \frac{h}{r_c} \left[ 1 + \frac{3}{4} \tan(\theta) - \frac{\pi/2 - \theta}{\cos^2(\theta)} \left( \frac{h}{r_c} \right) \right]
\]

(1.24)

Note that the correction term \( \frac{3}{4} \left[ \tan(\theta) - \frac{\pi/2 - \theta}{\cos^2(\theta)} \right] \left( \frac{h}{r_c} \right) \) is zero at \( \theta = \frac{\pi}{2} \).

We will employ linear approximation of \( \tan(\theta) - \frac{\pi/2 - \theta}{\cos^2(\theta)} \). Let \( \theta = \frac{\pi}{2} + \alpha \). Then

\[
\tan\left( \frac{\pi}{2} + \alpha \right) = -\frac{-\alpha}{\cos^2\left( \frac{\pi}{2} + \alpha \right)} \approx B\alpha
\]

(1.25)

We will now solve for coefficient \( B \). Algebra yields

\[
\sin\left( \frac{\pi}{2} + \alpha \right) \cos\left( \frac{\pi}{2} + \alpha \right) + \alpha \approx B\alpha \cos^2\left( \frac{\pi}{2} + \alpha \right)
\]

(1.26)

From trigonometry, we have

\[
\sin\left( \frac{\pi}{2} + \alpha \right) = \sin\left( \frac{\pi}{2} \right) \cos(\alpha) + \cos\left( \frac{\pi}{2} \right) \sin(\alpha)
\]

\[
\cos\left( \frac{\pi}{2} + \alpha \right) = \cos\left( \frac{\pi}{2} \right) \cos(\alpha) - \sin\left( \frac{\pi}{2} \right) \sin(\alpha)
\]

Therefore

\[- \cos(\alpha) \sin(\alpha) + \alpha \approx B\alpha \left[ - \sin(\alpha) \right]^2
\]

(1.27)

A Taylor series expansion yields
\[
\sin(\alpha) \approx \alpha - \frac{\alpha^3}{6} \quad \cos(\alpha) \approx 1 - \frac{\alpha^2}{2}
\]

and thus
\[
-(\alpha - \frac{\alpha^3}{6})(1 - \frac{\alpha^2}{2}) + \alpha \approx B\alpha[\alpha - \frac{\alpha^3}{6}]^2
\]  
(1.28)

or
\[
\frac{2\alpha^3}{3} \approx B\alpha^3
\]  
(1.29)

which allows us to solve for \(B\).

\[
B = \frac{2}{3}
\]

Using this value, we may find the linear approximation we seek

\[
\tan(\theta) - \frac{\pi}{2} - \frac{\theta}{\cos^2(\theta)} \approx \frac{2}{3}(\theta - \frac{\pi}{2})
\]  
(1.30)

Therefore, we may approximate function \(F\) as

\[
F_1 = \frac{1}{4}\{\tan(\theta) - \frac{\pi}{2} - \theta\} \approx \frac{1}{6}\{\theta - \frac{\pi}{2}\}
\]  
(1.31)

Substitution into (1.17) yields

\[
dS_{ht} \approx -\frac{\pi r_c^2}{h}[1 + \frac{1}{2}\{\theta - \frac{\pi}{2}\}(\frac{h}{r_c})]dh
\]  
(1.32)

Now, we turn our attention to the liquid-gas surface area \(S_{lg}\):

\[
S_{lg} = \int_0^h 2\pi r(z) \, dz = \int_0^1 2\pi[r_c + h f(\theta, \xi)]h \, d\xi = 2\pi r_c h + 2\pi h^2 \int_0^1 f(\theta, \xi) \, d\xi
\]  
(1.33)

Hence

\[
S_{lg} = 2\pi r_c h + 2\pi h^2 F_1
\]  
(1.34)
Taking a total derivative

\[ dS_{lg} = 2\pi h \, dr_c + [2\pi r_c + 4\pi h \, F_i] \, dh \]  \hspace{1cm} (I.35)

From Eqn. (I.14) we have

\[ dr_c \approx -\frac{r_c}{2h} \left[ 1 + 3 \left( \frac{h}{r_c} \right) F_i \right] \, dh \]  \hspace{1cm} (I.36)

Substitute into Eqn. (I.14) and simplifying yields

\[ dS_{lg} = \pi r_c \left[ 1 + \left( \frac{h}{r_c} \right) F_i \right] \, dh \]  \hspace{1cm} (I.37)

The first term on the right hand side represents the case of a cylinder \( F_i = 0 \).

Using the approximation \( F_i = \frac{1}{6} (\theta - \frac{\pi}{2}) \)

\[ dS_{lg} = \pi r_c \left[ 1 + \frac{1}{6} (\theta - \frac{\pi}{2}) \left( \frac{h}{r_c} \right) \right] \, dh \]  \hspace{1cm} (I.38)

Alternatively, we may approximate \( F_i \) in terms of \( \cos(\theta) \)

\[ \tan(\theta) - \frac{\pi}{2} - \theta \approx \frac{2}{3} (\theta - \frac{\pi}{2}) \]  \hspace{1cm} (I.39)

where \( \theta = \cos^{-1}(c) \). Using a Taylor series about \( \frac{\pi}{2} \) yields

\[ \cos^{-1}(c) = \cos^{-1}(0) + \frac{-1}{\sqrt{1 - \cos^2(\frac{\pi}{2})}} (c - \cos(\frac{\pi}{2})) \]  \hspace{1cm} (I.40)

Hence

\[ \theta \approx \frac{\pi}{2} - c \]

or
\[ \theta - \frac{\pi}{2} \approx -c \]

From this result we may obtain

\[ \tan(\theta) = \frac{\pi / 2 - \theta}{\cos^2(\theta)} \approx -\frac{2}{3} \cos(\theta) \]  

(I.41)

and the differential relations

\[ dS_{ls} \approx \frac{\pi r_c^2}{h} \left[ 1 - \frac{1}{2} \cos(\theta) \left( \frac{h}{r_c} \right) \right] dh \]  

(I.42)

\[ dS_{lg} \approx \pi r_c \left[ 1 - \frac{1}{6} \cos(\theta) \left( \frac{h}{r_c} \right) \right] dh \]  

(I.43)
References of Appendices:


