POD-Galerkin Projection of Flapping Wings

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Direct numerical simulation (DNS) and POD-Galerkin projection using a general pressure correction method are conducted for a flapping wing in pitching and plunging motion with a pitching amplitude of 10°, 20° and 30°. POD analysis shows that most turbulent kinetic energy (95%) is contained by the first four modes. The importance of pressure correction for Galerkin projection is discussed in terms of the reduced prediction error. It also shows the effectiveness of current POD-Galerkin projection method for developing reduced order models of two-dimensional flapping wings.

Nomenclature

\[ \Phi_i = \text{POD base element of velocity} \]
\[ \phi_{i,j} = \text{The } j\text{-th row of the } i\text{-th mode vector of velocity} \]
\[ \Psi_i = \text{POD base element of pressure gradient} \]
\[ \hat{E} = \text{Prediction error} \]
\[ U^i = \text{Velocity snapshot} \]
\[ \bar{U} = \text{Arithmetic average of velocity field of snapshots} \]
\[ u = \text{Exact velocity field} \]
\[ u_i = \text{Fluctuating velocity field} \]
\[ u_m = \text{Arithmetic average of velocity field of snapshots} \]
\[ \hat{u} = \text{Velocity field predicted by Galerkin projection} \]
\[ \bar{\nabla}p = \text{Arithmetic average of pressure gradient of snapshots} \]

I. Introduction

The design of micro-air vehicles (MAVs) is right on the road learning from birds and insects with flapping wings since MAVs have the similar size of these kinds of animals and their tasks for military and civic purposes require stability and maneuverability. The commons shared by animals and MAVs in aerodynamics and wing kinematics naturally make the vortex development dominant under low Reynolds number, to which many research works have been done for understanding the governing mechanisms in past decades but are not adequate enough to guide the MAV design. Direct numerical simulation (DNS) for flapping wing studies has proved its validity and provides an alternative which is accurate and costs less than the trial-and-error experimental approach. However, the time consumption of DNS is too enormous to be suitable for rapid development in current computational power, especially for three-dimensional applications. Thus, it is essential to have reduced-order models (ROMs), specifically proper orthogonal decomposition (POD) which can reconstruct unsteady physical features through energetic sense with given datasets of flow field that could be obtained from DNS results. Moreover, the ODEs

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projected from Galerkin projection of the Navier-Stokes equations for unsteady forces are ideal for control and design in parameter space.

The early applications of POD in fluid dynamics were to study the coherent structures in turbulent flow\(^\text{10}\). Before the importance of pressure term in the empirical Galerkin projection was aware and analyzed by Noack et al.\(^\text{11}\), the pressure term in projection equations was technologically omitted by setting specific flow boundaries\(^\text{12-15}\). It has been proposed in Ref. 11 that a pressure-Poisson equation which is derived from the incompressible Navier-Stokes equations is numerically solved to get the pressure field, and then the pressure gradient term is able to be expressed as the function of pressure. The main disadvantage of this method is that solving the pressure-Poisson equation is very time consuming and comparable to solving the Navier-Stokes equations\(^\text{15}\). It also has the same difficulty on getting a convergent solution. By introducing the POD of pressure field into the pressure-Poisson equation, Akhtar et al.\(^\text{16}\) studied the problem of flow past a cylinder. The forces coefficients from the projection of the pressure-Poisson equation and CFD simulations showed a good agreement. The validity of the POD of pressure field applied into the Galerkin projection of the Navier-Stokes equations is recently studied by Liang et al.\(^\text{17}\) in the flow past a stationary membrane plate with an angle of attack 30\(^\circ\). The results successfully show a great improvement in reducing the prediction error, compared to the method without considering pressure term.

To the best of our knowledge, there is lack of physics-based reduced-order models for flapping wing problems. As one of few successful attempts, Lewin and Harriri\(^\text{18}\) applied coordination transformation to a moving airfoil and performed the Galerkin projection to a modified momentum equation. The pressure term in this equation is therefore reduced to zero. Good projection results were obtained; however, the method is hard to be applied to 3D flapping wing problem and the flow with multiple body interaction or flexible wing due to the complexity of the derivation of the modified momentum equations.

In this paper, a general pressure correction method based on the POD of pressure field gradient for the Galerkin projection of the Navier-Stokes equations is presented for the application of flapping wings. It is examined first by the study of flow past a pitching-plunging plate with pitching amplitude of 30\(^\circ\). Then a comparison of cases with pitching amplitude 10\(^\circ\) and 20\(^\circ\) is performed. Comparing to Ref. 14, the derivation of the Galerkin projection in this work is much simpler and can be easily applied to 2D/3D flows with complex moving bodies.

II. Methodology

A. Direct Numerical Simulation

The non-dimensional incompressible Navier-Stokes equations, as written in Eq. (1)

\[
\frac{\partial u_i}{\partial x_i} = 0 ; \quad \frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j} u_i u_j = - \frac{\partial p}{\partial x_i} + \frac{1}{Re} \frac{\partial^2 u_i}{\partial x_j \partial x_j}
\]

are discretized on a Cartesian mesh using second-order central difference scheme in space. A second-order accurate fractional-step method for time advancement is employed. Its key feature is that simulations with complex boundaries can be carried out on stationary non-body conformal Cartesian grids, eliminating the need for complicated re-meshing algorithms that are usually employed with conventional Lagrangian body-conformal methods. The boundary conditions on the immersed body are imposed through a "ghost-cell" procedure\(^\text{17}\). The pressure-Poisson equation is solved using the geometric multi-grid method integrating with the immersed-boundary methodology. Detailed validations of the code can be found in Ref. 17.

B. Proper Orthogonal Decomposition (POD)

Proper Orthogonal Decomposition is a method used to represent large data fields with a relatively small number of elements. POD creates a set of basis function which spans the original data set by capturing the characteristic components. The system can be represented by the first few dominant modes.

Let \(\{U^i(x) : 1 \leq i \leq N, x \in \Omega\}\) be a set of N snapshots on domain \(\Omega\). The mean velocity of the snapshots is given by \(\bar{U} = \frac{1}{N} \sum_{i=1}^{N} U^i(x)\). Thus the new snapshots of the fluctuating velocity are \(u_i = U^i - \bar{U} , i = 1, \ldots, N\). It has been shown in previous work\(^\text{18}\) that an eigenvalue problem of a coefficient matrix \(A\) can be constructed using snapshots of \(u_i\) as

\[
AV = \lambda V
\]

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where the entry of matrix $A$ is $a_{ij} = \int_{\Omega} u_i(x) u_j(x) \, dx$. After solving the eigenvalue problem, the eigenvalue $\lambda_j$ and corresponding eigenvector $V_j$ are usually reorganized in descending order. Each POD basis elements, $\Phi_{ij}$, is given by $\Phi_{ij} = \sum_{j=1}^{N} a_{ij} u_j$ where $a_{ij}$ is the $j$-th element of the $i$-th eigenvector, $V_j$. Based on $\Phi_{ij}$, any snapshot can then be reconstructed using a linear combination of the POD basis elements $U'' = \tilde{U} + \sum_{i=1}^{N} \alpha_{ii} \Phi_i$ where $\alpha_{ii} = \Phi_i \cdot u_n$

C. Galerkin Projection

The basis function produced by POD is used to generate a predictive model via a Galerkin projection of the incompressible Navier-Stokes equations, Eq. (1). For the velocity field, the Galerkin approximation is

$$\left( \Phi_i \frac{\partial U}{\partial t} + (U \cdot \nabla) U \right) + \left( \Phi_i, \nabla p \right) = \frac{1}{\text{Re}} \left( \Phi_i, \nabla^2 U \right)$$

in which $(a, b) = \int_{\Omega} a \cdot b \, d\Omega$ denotes the inner product of two vectors over domain $\Omega$.

In current study, general velocity and pressure boundary conditions are employed in the direct numerical simulation, thus term $(\Phi_i, \nabla p)$ does not vanish and needs specific treatment. In previous study\textsuperscript{16}, the proper orthogonal decomposition of pressure field is done first. The pressure gradient is then constructed by calculating the gradient of the modes. However, the solution of pressure field for the problem using merely Neumann pressure boundary condition is a family of functions in the form of

$$p(x,t) = p_g(x) + p_a(t),$$

where $p(x,t)$ is the absolute pressure, $p_g(x)$ is the gauge pressure and $p_a(t)$ is the ambient pressure, which is difficult to be obtained from the solutions directly. In this paper, proper orthogonal decomposition is directly applied to the pressure gradient field (since $\nabla p(x,t) = \nabla p_g(x)$), which yields the following expression

$$\nabla p''(x,t) = \nabla p(x) + \sum_{j=1}^{N} \beta_j(t) \Psi_j(x)$$

where $\nabla p''(x,t)$ can be calculated from DNS.

Substituting Eq. (4) into Eq. (3) leads to the following set of evolution equations for the mode amplitude $a_i(t)$

$$\frac{d\alpha_i^n}{dt} = a_i + \sum_{j=1}^{N} \left( b_{ij} \alpha_j^n + d_{ij} \beta_j \right) + \sum_{j=1}^{N} \sum_{k=1}^{N} c_{ijk} \alpha_j^n \alpha_k$$

where

$$\begin{align*}
a_i &= -\left( \Phi_i, (\tilde{U} \cdot \nabla) \tilde{U} \right) - \left( \Phi_i, \nabla p \right) + \frac{1}{\text{Re}} \left( \Phi_i, \nabla^2 \tilde{U} \right) \\
b_{ij} &= -\left( \Phi_i, (\Phi_j \cdot \nabla) \tilde{U} \right) - \left( \Phi_i, (\tilde{U} \cdot \nabla) \Phi_j \right) + \frac{1}{\text{Re}} \left( \Phi_i, \nabla^2 \Phi_j \right) \\
c_{ijk} &= -\left( \Phi_i, (\Phi_j \cdot \nabla) \Phi_k \right) \\
d_{ij} &= -\left( \Phi_i, \Psi_j \right)
\end{align*}$$

No boundary integral is used since it will introduce the temporal effect into the coefficients of evolution equations (Eq. (6)).

As next step, similar to the strategy used in Ref. 15, the Galerkin projection of pressure-Poisson equation is
\[
(\Psi_i, \Delta p) = - \left( \Psi_i, \frac{\partial u_j}{\partial x_k} \frac{\partial u_k}{\partial x_j} \right)
\]

(7)

The Einstein notation is used in the right-hand side of Eq. (7). It can be deducted to the following form

\[
\sum_{j=1}^{N} e_{ij} \beta_j(t) = f_i + \sum_{j=1}^{N} g_{ij} \alpha_j(t) + \sum_{j=1}^{N} \sum_{k=1}^{N} h_{ijk} \alpha_j(t) \alpha_k(t)
\]

(8)

where

\[
e_{ij} = (\Psi_i, \nabla \cdot \Psi_j)
\]
\[
f_i = - \left( \Psi_i, \frac{\partial u_{m,k}}{\partial x_j} \frac{\partial u_{m,j}}{\partial x_k} \right) - \left( \Psi_i, \nabla \cdot \left( \nabla \rho \right) \right)
\]
\[
g_{ij} = -2 \left( \Psi_i, \frac{\partial u_{m,k}}{\partial x_j} \frac{\partial \phi_{j,r}}{\partial x_k} \right)
\]
\[
h_{ijk} = - \left( \Psi_i, \frac{\partial \phi_{j,s}}{\partial x_j} \frac{\partial \phi_{k,r}}{\partial x_k} \right)
\]

(9)

Given initial pressure gradient by numerically solving the Navier-Stokes equations or the pressure-Poisson equation, it is possible to solve two sets of equations (Eq. (5) and Eq. (8)). Current work only involves the Galerkin projection on the Navier-Stokes equation. Further investigation on the coupled system will be conducted in the future.

For later discussion, the prediction error defined in Ref. 12 is used:

\[
\hat{E} = \frac{(\mathbf{u} - \mathbf{u}, \mathbf{u} - \mathbf{\hat{u}})}{(\mathbf{u} - \mathbf{u}_m, \mathbf{u} - \mathbf{u}_m)}
\]

(10)

III. Results

A. Pitching and plunging wing at pitching amplitude of 30°

A 2-D membrane airfoil undergoing a combined motion of pitch-and-heave is placed in a flow field with right-travelling free stream of velocity \( U_\infty = 1 \). The velocity boundaries are of Dirichlet boundary condition except at right-hand boundary where it is of Neumann boundary of outflow. The pressure on all the four boundaries is of homogeneous Neumann boundary condition. The motion of membrane airfoil can be prescribed by

\[
h(t) = H \sin(2\pi ft - \phi)
\]
\[
\alpha(t) = A \sin(2\pi ft)
\]

(11)

(12)

where \( H \) is the heave amplitude, \( A \) is the amplitude of the sinusoidal pitch angle variation, \( f \) is the frequency, and \( \phi \) is the phase difference between the pitch and plunge motions. In current study, \( H \) is set to 0.5, \( A \) is 30°, \( \phi \) is \( \pi/2 \). Reynolds number \( Re = U_\infty c/\nu = 200 \) where \( c \) is the chord length of the wing of one unit length. Strouhal number \( St = 2Hf/U_\infty = 0.6 \).

Five datasets are extracted from a DNS simulation in which the vortex shedding has already reached a quasi-steady state at 960 frames per period. From the global data we pick out one snapshot every 1, 10 and 30 frames respectively. For better understanding the Galerkin projection method, we also pick out snapshots every 1 and 10 time frames in two periods. The case index is listed in Table 1. 32 modes are used for the POD and Galerkin projection of all the cases. The pressure term in Eq. (6) uses the proper orthogonal decomposition of pressure gradient directly.

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Table 1: Prediction Error

<table>
<thead>
<tr>
<th>Case Index</th>
<th>A</th>
<th>Dataset</th>
<th>Mean of Error</th>
<th>Standard deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>C1</td>
<td>30°</td>
<td>1 Period, 960 Frames</td>
<td>9.458E-02</td>
<td>0.1501</td>
</tr>
<tr>
<td>C2</td>
<td></td>
<td>1 Period, 96 Frames</td>
<td>2.061E-01</td>
<td>0.3661</td>
</tr>
<tr>
<td>C3</td>
<td></td>
<td>1 Period, 32 Frames</td>
<td>7.147E-01</td>
<td>1.4311</td>
</tr>
<tr>
<td>C4</td>
<td></td>
<td>2 Periods, 192 Frames</td>
<td>3.502E-01</td>
<td>0.3981</td>
</tr>
<tr>
<td>C5</td>
<td></td>
<td>2 Periods, 64 Frames</td>
<td>8.195E-01</td>
<td>0.8894</td>
</tr>
<tr>
<td>C6</td>
<td>20°</td>
<td>1 Period, 96 Frames</td>
<td>1.738E-01</td>
<td>0.2390</td>
</tr>
<tr>
<td>C7</td>
<td>10°</td>
<td>1 Period, 96 Frames</td>
<td>4.885E-01</td>
<td>1.3942</td>
</tr>
</tbody>
</table>

Figure 1 shows a comparison of vorticity contour at t/T=1 between DNS result and Galerkin projection result of the case C2 listed in Table 1. It shows the vortex structures in wake are accurately captured by current Galerkin projection method after shedding for one period. Although near the trailing edge of the flapping wing the shape of vortex at the upper surface is a little bit distorted and the vortex structure at lower surface is discontinues, the major feathers of the vortices are still clear and similar to the DNS result. The vorticity plot from the case C1 is also checked, however, it does not show obvious difference to Fig. 1(b).

The eigenvalue spectrum is shown in Fig. 2. First two modes contain nearly 90% of total turbulent kinetic energy. First four modes contain about 95% of the total. The accumulation of first six modes increases to nearly 98%. It is sufficient to check the amplitudes of the first four basis functions (Fig. 3). The curves of first four modes show a good match between predicted and projected results. The eigenvalue spectrums of the other cases for pitching amplitude 30° in Table 1 are almost identical to that of the case C2 except high order modes. This behavior shows a kind of convergence in the calculation of eigenvalue. The amplitudes of basis function of other cases are similar to C2.

The velocity contours for the mean flow and the first four modes are shown in Fig. 4. In the mean flow there is a strong jet in x-direction and no obvious flow in y-direction can be found in far field. Note the pairing of similar patterns in the contours of modes, shifted spatially, as a result of the convective nature of the flow. Mode 1 and mode 2 come in pairs and have the similar pattern, in both x and y directions, except a phase shift of π approximately. Mode 3 and mode 4 also come in pairs and have similar pattern with an approximate phase shift of π/2, in both x and y direction.
Figure 3. Time history of amplitude of basis functions

a), The first and second modes of velocity
b), The third and fourth modes

c), x-direction, mean velocity contours
d), y-direction, mean velocity contours
e), x-direction, mode 1
f), y-direction, mode 1
The effects of the number of snapshot and period on the POD-Galerkin projection method are also examined. Figure 5 shows the time history of the projection error. When constructing the entry of matrix $A$ in Eq. (2), the snapshots should be uncorrelated and linearly independent; however, the method we extract the snapshots determines that the uncorrelation between two snapshots which are temporally successive is deteriorated when the sampling frequency increases. Meanwhile, keeping the sampling frequency but increasing the sampling period generates similar sets of successive snapshots. It also deteriorates the uncorrelation of snapshots. So it is not surprised to see that the case of 2 periods, 192 frames has the global maximum projection error while the case of 1 period, 32 frames has the global minimum. Figure 6 shows the time history of the prediction error. Generally, the curves of the prediction error show a trend of increasing; especially the highest error appears near the end of time, similar to the result in Ref. 12. The mean and standard deviation of the prediction error are tabulated in Table 1. It can be found that the improvement in mean and standard deviation of the errors gained by increasing the number of snapshots is obvious; however, this can not be achieved by increasing the number of period. The effect of periodicity on the prediction error is complex. For case C5, the standard deviation is smaller than that of case C3. But it is not true for case C4 and case C2. Further investigation on it is still needed. A comparison of several methods is shown in Figure 7. Note that the dash line represents the proposed algorithm and the dash-dot line is the algorithms applying POD on pressure field which has a minimal enhancement comparing to the algorithm without pressure term correction. It is very obvious that current method has a mean error at least one order of magnitude smaller than the others.

B. Pitching and plunging wing at pitching amplitude of 10° and 20°

A parametric study on the pitching angle $\alpha$ of 10° and 20° is conducted. The flow boundary conditions and the controlling equations for wing kinematics are same for the DNS simulations. The datasets of 96 snapshots are extracted uniformly from 960 frames of one period. It shows that the eigenvalues can be group as pairs, especially...
for first six eigenvalues (Fig. 8). The energy decay of case 10° is faster than the other two cases after six eigenvalues. An integrated comparison of the prediction and projection error is shown in Fig. 9 and Table 1. It can be seen the prediction errors generally increase as time goes, however, the prediction errors of case 10° and 20° has the same order of magnitude of case 30°. This shows that the pressure correction method can be generally used for flapping plate.

Mean velocity and the first four modes of case 10° are shown in from Fig. 10. From these figures we can observe well-organized patterns, whose profile and intensity change according to the pitching angle. For instance, the area and strength of concentrated zone of turbulent kinetic energy near the leading edge of flapping wing in x-direction of velocity mode 1 enhance as the pitch angle decreases.
Figure 10. Mean velocity contours and POD modes for pitching amplitude of 10°.
IV. Conclusion

A general pressure correction method for the POD-Galerkin projection on the Navier-Stokes equations is proposed for flow fields with moving boundaries in this paper. It is studied by the flow past a flat plate in pitching and plunging motion with pitching amplitude changing from 10° to 30°. The prediction error of the studied cases is smaller than the methods which involves no pressure correction or pressure correction merely with the POD of pressure. In these cases, the first four modes capture approximately 95% of turbulent kinetic energy. It also shows that certain sorts of similar patterns in POD modes among these cases. Application of this method to flapping tandem wings and flexible wings are still ongoing.

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