RISK, RETURN AND EQUILIBRIUM: SOME CLARIFYING COMMENTS

EUGENE F. FAMA*

SHARPE [12] AND LINTNER [7] have recently proposed models directed at the following questions: (a) What is the appropriate measure of the risk of a capital asset? (b) What is the equilibrium relationship between this measure of the asset's risk and its one-period expected return? Lintner contends that the measure of risk derived from his model is different and more general than that proposed by Sharpe. In his reply to Lintner, Sharpe [13] agrees that their results are in some ways conflicting and that Lintner's paper supersedes his.

This paper will show that in fact there is no conflict between the Sharpe-Lintner models. Properly interpreted they lead to the same measure of the risk of an individual asset and to the same relationship between an asset's risk and its one-period expected return. The apparent conflicts discussed by Sharpe and Lintner are caused by Sharpe's concentration on a special stochastic process for describing returns that is not necessarily implied by his asset pricing model. When applied to the more general stochastic processes that Lintner treats, Sharpe's model leads directly to Lintner's conclusions.

I. EQUILIBRIUM IN THE SHARPE MODEL

The Sharpe capital asset pricing model is based on the following assumptions:

(a) The market for capital assets is composed of risk averting investors, all of whom are one-period expected-utility-of-terminal-wealth maximizers (in the von Neumann-Morgenstern [16] sense) and find it possible to make optimal portfolio decisions solely on the basis of the means and standard deviations of the probability distributions of terminal wealth associated with the various available portfolios.2 If the one-period return on an asset or portfolio is defined as the change in wealth during the horizon period divided by the initial wealth invested in the asset or portfolio, then the assumption implies

* Associate Professor of Finance, Graduate School of Business, University of Chicago. In preparing this paper I have benefited from discussions with members of the Workshop in Finance of the Graduate School of Business. The comments of M. Blume, P. Brown, M. Jensen, M. Miller, H. Roberts, R. Roll, M. Scholes and A. Zellner were especially helpful. The research was supported by a grant from the Ford Foundation.

1. The terms "capital asset" and "one-period return" will be defined below.

2. In the one-period expected utility of terminal wealth model, the objects of choice for the investor are the probability distributions of terminal wealth provided by each asset and portfolio. Each "portfolio" represents a complete investment strategy covering all assets (e.g., bonds, stocks, insurance, real estate, etc.) that could possibly affect the investor's terminal wealth. That is, at the beginning of the horizon period the investor makes a single portfolio decision concerning the allocation of his investable wealth among the available terminal wealth producing assets. All terminal wealth producing assets are called capital assets.
that investors can make optimal portfolio decisions on the basis of means and standard deviations of distributions of one-period portfolio returns.\(^8\)

(b) All investors have the same decision horizon, and over this common horizon period the means and variances of the distributions of one-period returns on assets and portfolios exist.

(c) Capital markets are perfect in the sense that all assets are infinitely divisible, there are no transactions costs or taxes, information is costless and available to everybody, and borrowing and lending rates are equal to each other and the same for all investors.

(d) Expectations and portfolio opportunities are "homogenous" throughout the market. That is, all investors have the same set of portfolio opportunities, and view the expected returns and standard deviations of return\(^4\) provided by the various portfolios in the same way.

Assumption (a) places the analysis within the framework of the Markowitz \([10]\) one-period mean-standard deviation portfolio model. Tobin \([15]\) shows that the mean-standard deviation framework is appropriate either when probability distributions of portfolio returns are normal or Gaussian\(^6\) or when investor utility of return functions are well-approximated by quadratics. In either case the optimal portfolio for a risk averter will be a member of the mean-standard deviation efficient set, where an efficient portfolio must satisfy the following criteria: (1) If any other portfolio provides lower standard deviation of one-period return, it must also have lower expected return; and (2) if any other portfolio has greater expected return, it must also have greater standard deviation of return.\(^6\)

3. The one-period return defined in this way is just a linear transformation of the units in which terminal wealth is measured; an investor's utility function can be defined in terms of one-period return just as well as in terms of terminal wealth. Note that the one-period return involves no compounding; it is just the ratio of the change in terminal wealth to initial wealth, even though the horizon period may be very long.

Though the remainder of the analysis will be in terms of the one-period return, we should keep in mind that the objects being priced in the market are the probability distributions of terminal wealth associated with each of the available capital assets.

4. Henceforth the terms "return" and "one-period return" will be used synonymously.

5. Tobin claims (and properly so) that the mean-standard deviation framework is appropriate whenever distributions of returns on all assets and portfolios are of the same type and can be fully described by two parameters. If the distribution of the return on a portfolio is always of the same type as the distributions of the returns on the individual assets in the portfolio, then that distribution must be a member of the stable (or stable Paretoian) class. But the only stable distribution whose variance exists is the normal.

6. The mean-standard deviation model presupposes, of course, the existence of means and variances for all distributions of one-period returns. The work of Mandelbrot \([9]\), Fama \([2]\), and Roll \([11]\) suggests, however, that this assumption may be inappropriate, at least with respect to the standard deviation. Distributions of returns on common stocks and bonds apparently conform better to members of the stable or stable Paretnian family for which the variance does not exist than to the normal distribution (the only member of the family for which the variance does exist). This does not mean that mean-standard deviation portfolio models are useless. Fama \([3]\) has shown that insights into the effects of diversification on dispersion of return that are derived from the mean-standard deviation model remain valid when the model is generalized to include the entire stable family. In a later paper \([4]\) it is shown that much of the Sharpe-Lintner mean-standard deviation capital asset pricing model can also be generalized to include the non-normal members of the stable family. Thus it is not inappropriate to reconsider the Sharpe-Lintner models, since resolution of the apparent conflicts between them has implications for the more general model of \([4]\).
Assumptions (b), (c) and (d) of the Sharpe model standardize the picture of the portfolio opportunity set available to each investor. Assumption (b) implies that the portfolio decisions of all investors are made at the same point in time, and the horizon considered in making these decisions is the same for all. Assumptions (c) and (d) standardize both the set of available portfolios and investors' evaluations of the combinations of expected return and standard deviation provided by each member of the set.

The situation facing each investor can be represented as in Figure 1. The horizontal axis of the figure measures expected return $E(R)$ over the common horizon period, while the vertical axis measures standard deviation of return, $\sigma(R)$. If attention is restricted to portfolios involving only risky assets, Sharpe [12] shows that the set of mean-standard deviation efficient portfolios will fall along a curve convex to the origin, like LMO in Figure 1.

7. Lintner [7, pp. 600-01] considers an extension of the asset pricing model to the case where investors disagree on the expected returns and standard deviations provided by portfolios. The results are essentially the same as those derived under the assumption of "homogenous expectations." Since Lintner's criticism of Sharpe does not depend on this part of his work, our discussion will use the simpler "homogeneous expectations" version of the model. Most of Lintner's discussion is also within this framework, and in all other respects his assumptions are identical to those of Sharpe.

It is important to emphasize that the Sharpe-Lintner asset pricing models, like the Markowitz-Tobin portfolio models, present one-period analyses. For a more complete discussion of the one-period framework see [4].

8. Strictly speaking this result presupposes that there are at least two portfolios in the efficient set. That is, there is no portfolio which has both higher expected return and lower standard devia-
The model assumes, however, that in addition to the opportunities presented by portfolios of risky assets, there is a riskless asset $F$ which will provide the sure return $R_F$ over the common horizon period; it is assumed that the investor can both borrow and lend at the riskless rate $R_F$. Consider portfolios $C$ involving combinations of the riskless asset $F$ and an arbitrary portfolio $A$ of risky assets. The expected return and standard deviation of return provided by such combinations are

\begin{align*}
E(R_C) &= xR_F + (1 - x)E(R_A) \quad x \leq 1, \\
\sigma(R_C) &= (1 - x)\sigma(R_A),
\end{align*}

where $x$ is the proportion of available funds invested in the riskless asset $F$, so that $(1-x)$ is invested in $A$. Applying the chain rule,

\begin{align*}
\frac{d}{dx}\frac{\sigma(R)}{E(R)} &= \frac{d\sigma(R)}{dx} \cdot \frac{dx}{dE(R)} = \frac{\sigma(R_A)}{E(R_A) - R_F}.
\end{align*}

This implies that the combinations of expected return and standard deviation provided by portfolios involving $F$ and $A$ must fall along a straight line through $R_F$ and $A$ in Figure 1.

It is now easy to determine the effects of borrowing-lending opportunities on the set of efficient portfolios. In Figure 1 consider the line $RFMZ$, touching $LMO$ at $M$. This line represents the combinations of expected return and standard deviation associated with portfolios where the proportion $x$ ($x \leq 1$) is invested in the riskless asset $F$ and $1-x$ in the portfolio of risky assets $M$. At the point $R_F$, $x = 1$, while at the point $M$, $x = 0$. Points below $M$ along $RFMZ$ correspond to lending portfolios ($x > 0$), while points above $M$ correspond to borrowing portfolios ($x < 0$). At given levels of $\sigma(R)$ there are portfolios along $RFMZ$ which provide higher levels of $E(R)$ than the corresponding portfolios along $LMO$. Thus (except for $M$) the portfolios along $LMO$ are dominated by portfolios along $RFMZ$, which is now the efficient set.

The conditions necessary for equilibrium in the asset market can now be stated. Since all investors have the same horizon and view their portfolio opportunities in the same way, the Sharpe model implies that everybody faces the same picture of the set of efficient portfolios. If the relevant picture is Figure 1, then all efficient portfolios for all investors will lie along $RFMZ$. More risky efficient portfolios involve borrowing ($x < 0$) and investing all available funds (including borrowings) in the risky combination $M$. Less risky efficient portfolios involve lending ($x > 0$) some funds at the rate $R_F$ and investing remaining funds in $M$. The particular portfolio that an investor chooses will depend on his attitudes toward risk and return, but optimum portfolios for all investors will involve some combination of the riskless asset $F$ and the portfolio of risky assets $M$. There will be no incentive for anyone

9. As noted earlier, Tobin [15] shows that the mean-standard deviation portfolio model is appropriate either when probability distributions of returns on portfolios are normal or when investor utility of return functions are well approximated by quadratics. In either case the indifference curves (i.e., loci of constant expected utility) of a risk averter will be positively sloping
to hold risky assets not included in M. If M does not contain all the risky
assets in the market, or if it does not contain them in exactly the proportions
in which they are outstanding, then there will be some assets that no one will
hold. This is inconsistent with equilibrium, since in equilibrium all assets must
be held.

Thus, if Figure 1 is to represent equilibrium, M must be the market port-
folio; that is, M consists of all risky assets in the market, each weighted by
the ratio of its total market value to the total market value of all assets. In
addition, the riskless rate \( R_F \) must be such that net borrowing in the market
is 0; that is, at the rate \( R_F \) the total quantity of funds that people want to
borrow is equal to the quantity that others want to lend.

As a description of reality, this view of equilibrium has an obvious short-
coming. In particular, all investors hold only combinations of the riskless
asset \( F \) and M. The market portfolio M is the only efficient portfolio of all
risky assets. This result follows from the assumed existence of the oppor-
tunity curve, which is linear, and concave to the origin in the \( E(R), \sigma(R) \) plane of Figure 1, with expected utility increasing
as we move on to indifference curves further to the right in the plane. Since the efficient set of portfo-
lios is linear, equilibrium for the investor (i.e., the point of maximum attainable expected
utility) will occur at a point of tangency between an indifference curve and the efficient set or
at the point \( R_p \). The degree of the investor's risk aversion will determine whether this will be
a point above or below M along \( R_p M Z \) in Figure 1.

10. Figure 1 itself does not tell us that the market portfolio M is the only combination of
risky assets with expected return and standard deviation \( E(R_M) \) and \( \sigma(R_M) \). Suppose there is
another portfolio \( G \) such that \( E(R_G) = E(R_M) \) and \( \sigma(R_G) = \sigma(R_M) \). Consider portfolios \( C \) where
the proportion \( x, (0, <x<1) \), is invested in \( G \) and \( (1-x) \) in M. Then
\[
E(R_C) = xE(R_G) + (1-x)E(R_M) = E(R_M)
\]
\[
\sigma(R_C) = \sqrt{x^2\sigma^2(R_G) + (1-x)^2\sigma^2(R_M) + 2x(1-x)\text{corr}(R_G, R_M)\sigma(R_G)\sigma(R_M)}^{1/2}
\]

It follows that \( \sigma(R_C) < \sigma(R_M) \) unless \( \text{corr}(R_G, R_M) = 1 \), that is, unless the returns on portfolios
\( G \) and M are perfectly correlated. The condition \( \sigma(R_C) < \sigma(R_M) \) is inconsistent with equilibrium,
since in equilibrium M must be a member of the efficient set. Thus, if there is a portfolio with
the same expected return and standard deviation as the market portfolio M, its returns must be
perfectly correlated with those of M, an unlikely situation. In any case, such a portfolio would
be a perfect substitute for M.

11. Sharpe [12] himself proposes a slightly different version of equilibrium, one which does
not imply that the market portfolio M is the only efficient portfolio of risky assets. He argues that
in equilibrium an entire segment of the right boundary of the set of feasible risky portfolios may
be tangent to a straight line through \( R_p \). He further shows that the returns on all portfolios
along such a segment must be perfectly correlated. Since ex post returns on portfolios of different
risky assets are never perfectly correlated, it is unlikely that investors will expect them to be
perfectly correlated ex ante, and so multiple tangencies would seem to represent an uninteresting
case. Note, though, that if a segment of the right boundary of the set of feasible risky portfolios is
tangent to a line through \( R_p \), to be consistent with equilibrium the market portfolio M must be
one of the tangency points along the segment. This is an implication of the fact that when the
portfolios of individuals are aggregated, the aggregate is just the market portfolio with zero net
borrowing. Thus, it must be possible to obtain the market portfolio by taking weighted combina-
tions of portfolios along the tangency segment.

In sum, given the assumptions of the Sharpe model, equilibrium can be associated (a) with a
situation where the market portfolio is the only efficient combination of risky assets or (b) with a
situation where there are many efficient combinations of risky assets, one of which is the market
portfolio. Fortunately, Sharpe shows that in using the portfolio model to develop the relationship
between risk and expected return on individual assets, it does not matter which of these represen-
tations of equilibrium is adopted. Because it simplifies the exposition of the model and also
seems to be more realistic, we shall concentrate on the case where the market portfolio is the
only efficient combination of risky assets. This is also the case dealt with by Lintner [7].
portunity to borrow or lend indefinitely at the riskless rate $R_F$. Fortunately, in
[4] it is shown that the measure of the risk of an individual asset and the
equilibrium relationship between risk and expected return derived from the
capital asset pricing model will be essentially the same whether or not it is
assumed that such riskless borrowing-lending opportunities exist.

II. THE MEASUREMENT OF RISK AND THE RELATIONSHIP
BETWEEN RISK AND RETURN

We consider now the major problems of the Sharpe capital asset pricing
model; that is, (a) determination of a measure of risk consistent with the
portfolio and expected utility models, and (b) derivation of the equilibrium
relationship between risk and expected return. It is important to note that the
development of the Sharpe model to this point is completely consistent with
Lintner [7]. In particular, the two models are based on the same set of
assumptions, and the resulting views of equilibrium are the same. Thus it
seems unlikely that the implications of the two models for the measurement
of risk and the relationship between risk and return can be different. In fact
it will now be shown that Sharpe’s approach leads to exactly the same con-
cclusions as Lintner’s. The “conflicts” which they find in their respective re-
results will be shown to arise from the fact that both misinterpret the implic-
atious of the Sharpe model.

For any risky asset $i$ there will be a curve, like $i$ $M$ $i'$ in Figure 1, which
shows the combinations of $E(R)$ and $\sigma(R)$ that can be attained by forming
portfolios of asset $i$ and the market portfolio $M$. If $x$ is the proportion of
available funds invested in asset $i$, the returns on such portfolios $(C)$ can be
expressed as

$$R_C = x R_i + (1 - x) R_M \quad (x \leq 1). \quad (4)$$

Now consider portfolios $(D)$ where the proportion $x$ is invested in the riskless
asset $F$ and $(1 - x)$ in the market portfolio $M$. The returns on such portfolios
will be given by

$$R_D = x R_F + (1 - x) R_M. \quad (5)$$

As noted earlier, the combinations of expected return and standard deviation
of return provided by such portfolios fall along the efficient set line $R_F M Z$ in
Figure 1. It is easy to show that the functions underlying $i$ $M$ $i'$ and LMO are
both differentiable at the point $M$. Since $R_F M Z$ is the efficient set, $i$ $M$ $i'$
and LMO must be tangent at $M$. That is,

$$\frac{d \sigma(R_C)}{d E(R_C)} = \frac{d \sigma(R_D)}{d E(R_D)}, \text{ when } x = 0. \quad (6)$$

The economic interpretation of (6) is familiar. $d \sigma(R_D)/d E(R_D)$ is the

12. When $0 \leq x \leq 1$ portfolios along $i$ $M$ $i'$ between $i$ and $M$ are obtained. At $x = 0$, the
market portfolio $M$ is obtained. Since $M$ contains asset $i$, even when $x = 0$ the portfolio $C$
will contain some of $i$. When $x < 0$, so that there is a short position in asset $i$, portfolios along the
segment $M$ $i'$ are obtained.

Though the discussion in the text is phrased in terms of individual assets, the analysis applies
directly to the case where $i$ is a portfolio.
marginal rate of exchange of standard deviation for expected return along the efficient set $R_X$ in $M_Z$. Since all investors have the same view of the efficient set, $d \sigma(R_D)/d E(R_D)$ is in fact the market rate of exchange. On the other hand, $d \sigma(R_C)/d E(R_C)$ is the marginal rate of exchange of standard deviation for expected return in the market portfolio as the proportion of asset $i$ in the market portfolio is changed. In equilibrium excess demand for asset $i$ must be 0. But this will only be the case if when $x = 0$ in (4), the expected return on asset $i$ is such that the marginal rate of exchange $d \sigma(R_C)/d E(R_C)$ is equal to the market rate of exchange $d \sigma(R_D)/d E(R_D)$.

Sharpe’s insight was in noting that the equilibrium condition (6) implies both a measure of the risk of asset $i$ and the equilibrium relationship between the risk and the expected return on the asset. Using the chain rule to derive expressions for $d \sigma(R_C)/d E(R_C)$ and $d \sigma(R_D)/d E(R_D)$, and then evaluating these derivatives at $x = 0$, (6) becomes

$$\frac{\text{cov}(R_i, R_M) - \sigma^2(R_M)}{[E(R_i) - E(R_M)] \sigma(R_M)} = \frac{\sigma(R_M)}{E(R_M) - R_F}. \tag{7}$$

To get an expression for the expected return on asset $i$, it suffices to solve (7) for $E(R_i)$, leading to

$$E(R_i) = R_F + \frac{[E(R_M) - R_F]}{\sigma^2(R_M)} \text{cov}(R_i, R_M), \quad i = 1, 2, \ldots, N, \tag{8}$$

where $N$ is the total number of assets in the market. Alternatively, the “risk premium” in the expected return on asset $i$ is

$$E(R_i) - R_F = \left[\frac{E(R_M) - R_F}{\sigma^2(R_M)}\right] \text{cov}(R_i, R_M) = \lambda \text{cov}(R_i, R_M), \quad i = 1, 2, \ldots, N. \tag{9}$$

Now (9) applies to each of the $N$ assets in the market, and the value of $\lambda$, the ratio of the risk premium in the expected return on the market portfolio to the variance of this return, will be the same for all assets. Thus the differences between the risk premiums on different assets depend entirely on the covariance term in (9). The coefficient $\lambda$ can be thought of as the market price per unit of risk so that the appropriate measure of the risk of asset $i$ is $\text{cov}(R_i, R_M)$. Thus this term certainly deserves closer study. In the process we shall find that (9), which is just a rearrangement of the last expression in Sharpe’s [12] footnote 22, is exactly Lintner’s [7] expression for the risk premium.

Note that by definition $R_M$, the return on the market portfolio, is just the weighted average of the returns on all the individual assets in the market. That is,

$$R_M = \sum_{j=1}^{N} X_j R_j, \tag{10}$$

13. That is,

$$\frac{d \sigma(R_C)}{d E(R_C)} = \frac{d \sigma(R_C)}{d x} \cdot \frac{dx}{d E(R_C)} \quad \text{and} \quad \frac{d \sigma(R_D)}{d E(R_D)} = \frac{d \sigma(R_D)}{d x} \cdot \frac{dx}{d E(R_D)}.$$
where $X_j$ is the proportion of the total market value of all assets that is accounted for by asset $j$. It follows that

$$
cov(R_t, R_M) = E\left(\left[R_M - E(R_M)\right] \left[R_t - E(R_t)\right]\right)
$$

$$
= E\left(\sum_{j=1}^{N} X_j \left[R_j - E(R_j)\right] \left[R_t - E(R_t)\right]\right)
$$

$$
= \sum_{j=1}^{N} X_j \text{cov}(R_j, R_t).
$$

Substituting (11) into (9) yields

$$
E(R_i) - R_F = \sum_{j=1}^{N} X_j \text{cov}(R_j, R_t), \quad i = 1, 2, \ldots, N,
$$

which is exactly Lintner's [7, p. 596] equation (11) but derived from Sharpe's model. 14

Within the context of the Sharpe model (12) is quite reasonable. From (10)

$$
\sigma^2(R_M) = \sum_{k=1}^{N} \sum_{j=1}^{N} X_k X_j \text{cov}(R_j, R_k) = \sum_{k=1}^{N} X_k \sum_{j=1}^{N} X_j \text{cov}(R_j, R_k).
$$

Now the term for $k = i$ in (13) is just

$$
X_i \sum_{j=1}^{N} X_j \text{cov}(R_j, R_i) = X_i \text{cov}(R_i, R_M).
$$

Thus $X_i \text{cov}(R_i, R_M)$ measures the contribution of asset $i$ to the variance of the return on the market portfolio. Since this contribution is proportional to $\text{cov}(R_i, R_M)$ and since the market portfolio is the only stochastic component in all efficient portfolios, it is not unreasonable that the risk premium on asset $i$ is proportional to $\text{cov}(R_i, R_M)$.

Note that (9) and (12) allow us to rank the risk premiums in the expected returns on different assets, but they provide no information about the magnitudes of the premiums. These depend on the difference $E(R_M) - R_F$, which in turn depends on the attitudes of all the different investors in the market toward risk and return. Without knowing more about attitudes toward risk, all we can say is that $E(R_M) - R_F$ must be such that in equilibrium all risky assets are held and the borrowing-lending market is cleared.

Thus, properly interpreted, the models of Sharpe and Lintner lead to identical conclusions concerning the appropriate measure of the risk of an individual asset and the equilibrium relationship between the risk of the

14. Lintner [7, 8] makes much of the fact that

$$
cov(R_i, R_M) = \sum_{j \neq i} X_j \text{cov}(R_j, R_i) + X_i \sigma^2(R_i)
$$

contains a term for the variance of asset $i$. He stresses the importance of the variance term in empirical studies concerned with measuring the riskiness of an individual asset. Recall, however, that $X_i$ is the total market value of all outstanding units of asset $i$ divided by the total market value of all assets. Thus the variance term in (14) is likely to be trivial relative to the weighted sum of covariances—a familiar result in portfolio models.
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asset and its expected return. What, then, is the source of the "conflict" between the two models which both authors apparently feel exists? Unfortunately Sharpe puts the major results of his paper in his footnote 22 [12, p. 438]; in the text he concentrates on applying these results to the market or "diagonal" model of the behavior of asset returns which he proposed in an earlier paper [14]. But the market model that he uses contains inconsistent constraints which lead to misinterpretation of the capital asset pricing model. Lintner, in his turn, does not appreciate the generality of Sharpe's results, and accepts (and in some ways misinterprets) Sharpe's treatment of the market model.

III. THE RELATIONSHIP BETWEEN RISK AND RETURN IN THE MARKET MODEL

In the "market model" which Sharpe [12, pp. 438-42] uses to illustrate his asset pricing model, it is assumed that there is a linear relationship between the one-period return on an individual asset and the return on the market portfolio M. That is,

$$R_i = a_i + \beta_i R_M + \epsilon_i \quad i = 1, 2, \ldots, N,$$

(15)

where $a_i$ and $\beta_i$ are parameters specific to asset $i$. It is further assumed that the random disturbances $\epsilon_i$ have the properties,

$$E(\epsilon_i) = 0 \quad i = 1, 2, \ldots, N \quad (16a)$$

$$\text{cov}(\epsilon_i, \epsilon_j) = 0 \quad i, j = 1, 2, \ldots, N; \quad i \neq j \quad (16b)$$

$$\text{cov}(\epsilon_i, R_M) = 0 \quad i = 1, 2, \ldots, N. \quad (16c)$$

Thus the assumption is that the only relationships between the returns on individual risky assets arise from the fact that the return on each is related to the return on the market portfolio $M$ via (15).

Applying the market model of (15) and (16) to the equivalent risk premium expressions (9) and (12) will allow us to pinpoint the apparent source of conflict between the results of Sharpe and Lintner. From (15) and (16)

$$\text{cov}(R_i, R_M) = E\{(\beta_i [R_M - E(R_M)] + \epsilon_i)(R_M - E(R_M))\} \quad (17a)$$

$$= \beta_i \sigma^2(R_M) + \text{cov}(\epsilon_i, R_M) \quad (17b)$$

$$= \beta_i \sigma^2(R_M). \quad (17c)$$

Substituting (17c) into (9) yields

$$E(R_i) - R_F = \lambda \beta_i \sigma^2(R_M) = [E(R_M) - R_F]\beta_i, \quad i = 1, 2, \ldots, N. \quad (18)$$

Thus when the stochastic process generating returns is as described by the market model of (15) and (16), the risk premium in the expected return on a given asset is proportional to the slope coefficient $\beta$ for that asset. The more sensitive the asset is to the return on the market portfolio, the larger its risk premium.

In discussing the implications of his capital asset pricing model Sharpe concentrates on (18). But it is important to remember that the market model assumes a very special stochastic process for asset returns which was not
assumed in the derivation of the general expressions (9) and (12) for the risk premium in the capital asset pricing model. The asset pricing model itself, as summarized by expressions (9) and (12), applies to much more general stochastic processes than those assumed in the market model and thus in (18). This point is especially crucial since we shall now see that the market model, as defined by (15) and (16), is inconsistent.

Expression (18) was obtained by applying the market model to (9). Since (12) and (9) are equivalent expressions for the risk premium in the expected return on asset i, it should be possible to apply the market model to (12) and obtain (18):

\[ E(R_i) - R_F = \lambda \sum_{j=1}^{N} X_j \text{cov}(R_j, R_i) \]

\[ = \lambda \left\{ \beta_1 \sum_{j=1}^{N} X_j \beta_j \sigma^2(R_M) + X_i \sigma^2(\varepsilon_i) \right\}, \]

which is exactly Lintner's [7, p. 605] expression (24). It will presently be shown that the market model implies \( \sum X_j \beta_j = 1 \). Thus (20) reduces to

\[ E(R_i) - R_F = \lambda [\beta_1 \sigma^2(R_M) + X_i \sigma^2(\varepsilon_i)] \]

or

\[ E(R_i) - R_F = \left[ E(R_M) - R_F \right] \left[ \beta_1 + \frac{X_i \sigma^2(\varepsilon_i)}{\sigma^2(R_M)} \right]. \]

But (22) includes a term involving \( \sigma^2(\varepsilon_i) \) which does not appear in (18), and this is the major source of controversy between Lintner and Sharpe. In applying the asset pricing model to the market model, Sharpe arrives at (18) while Lintner derives (20) or its equivalent (22). Lintner [7, pp. 607-08] presumes that Sharpe is considering the case where all residual variances \([the \sigma^2(\varepsilon_i)]\) are 0. But Sharpe clearly did not intend to impose this restriction on his model.\[^{18}\] In addition, (18) is derived directly from (9), (15), and (16), and there is no presumption in the derivation that the residual variances are 0.

In fact the discrepancy between (18) and (22) arises from an inconsistency in the specification of the market model; neither of these expressions for the risk premium is correct. Note that (10) and (15) together imply

\[ R_M = \sum_{j=1}^{N} X_j R_j = \sum_{j=1}^{N} X_j [\varepsilon_j + \beta_j R_M + \varepsilon_j]. \]

Thus, since \( \varepsilon_i \) is one of the terms in \( R_M \), (16c) is inconsistent with the remaining assumptions of the market model. Since (16c) is used in deriving both (18) and (22), these are both incorrect expressions for the risk premium in the market model.

\[^{15}\]Cf., Sharpe [12, pp. 438-39]. "The response of \( R_i \) to changes in \( R_g \) (our \( R_M \)) (and variations in \( R_g \) itself) account for much of the variation of \( R_i \). It is this component of the asset's total risk which we term the *systematic* risk. The remainder, being uncorrelated with \( R_g \), is the *unsystematic component.*" Though Sharpe does not explicitly specify the version of the market model he is considering, it seems clear from this quotation and the remainder of his discussion that, for his purposes, (15) and (16) represent the relevant model.
Unfortunately, (16c) is not the only inconsistency in the market model of (15), (16) and (23); it is also easy to show that (15), (16b) and (23) cannot hold simultaneously. Recalling that αj and βj are constants, (23) implies

\[ \sum_{j=1}^{N} X_j \alpha_j = 0, \quad \sum_{j=1}^{N} X_j \beta_j = 1, \]  

(24a)

\[ \sum_{j=1}^{N} X_j \epsilon_j = 0. \]  

(24b)

The constraints of (24a) pose no problem; (24b), however, is inconsistent with (16b)—we cannot assume that the disturbances are independent and then constrain their weighted sum to be 0.

One possible specification of the market model which does not lead to the problems discussed above is as follows.

\[ \mathbb{E}(e_i) = 0 \quad i = 1, 2, \ldots, N; \]  

(25a)

\[ \text{cov}(e_i, e_j) = 0 \quad i, j = 1, 2, \ldots, N; \quad i \neq j; \]  

(25b)

\[ \text{cov}(e_i, r_M) = 0 \quad i = 1, 2, \ldots, N. \]  

(25c)

In this model \( r_M \) is interpreted as a common underlying market factor which affects the returns on all assets. The relationship between \( r_M \) and the return on the market portfolio is then

\[ r_M = \sum_{j=1}^{N} X_j R_j = \sum_{j=1}^{N} X_j [\alpha_j + \beta_j r_M + \epsilon_j]. \]  

(26)

From either (9) or (12) it follows that in this model the risk premium on asset \( i \) is

\[ \mathbb{E}(R_i) - R_F = \lambda \sum_{j=1}^{N} X_j \text{cov}(R_j, R_i) \]

\[ = \lambda \sum_{j=1}^{N} X_j \mathbb{E}\{ (\beta_j [r_M - \mathbb{E}(r_M)] + \epsilon_j) (\beta_i [r_M - \mathbb{E}(r_M)] + \epsilon_i) \} \]

\[ = \lambda \left\{ \beta_i \sum_{j=1}^{N} X_j \beta_j \sigma^2(r_M) + X_i \sigma^2(\epsilon_i) \right\} \]  

(27)

which is equivalent to Lintner’s [7, equation (23)] expression for the risk premium in this more general version of the market model. But it is again important to note that Lintner’s results follow directly from (9) and (12), the general expressions for the risk premium developed in Sharpe’s model.

Finally, though the issues discussed above are certainly interesting from a theoretical viewpoint, from a practical viewpoint (18), (22) and (27) are nearly equivalent expressions for the risk premium in the market model. The empirical evidence of King [6] and Blume [1] suggests that, on average,
The size of the residual term in (22) will be determined primarily by \( X_i \), the proportion of the total value of all assets accounted for by asset \( i \), which will usually be quite small relative to \( \beta_i \) (which is on average 1). The risk premiums given by (18) and (22), then, will be nearly equal.

Next note that it is always possible to scale \( r_M \) in (26) so that \( \sum X_j a_j = 0 \) and \( \sum X_j \beta_j = 1 \). Then

\[
\sigma^2(r_M) = \sigma^2(X) + \sum_{j=1}^{N} X_j^2 \sigma^2(e_j).
\]

But again the weighted sum of residual variances will be small relative to \( \sigma^2(r_M) \) so that \( \sigma^2(r_M) \approx \sigma^2_X + \sigma^2_e \), which implies that the risk premiums given by (22) and (27) are almost equal.

### IV. Conclusions

In sum, then, there are no real conflicts between the capital asset pricing models of Sharpe [12] and Lintner [7, 8]. When they apply their general results to the market model, both make errors which turn out to be unimportant from a practical viewpoint. The important point is that their general models represent equivalent approaches to the problem of capital asset pricing under uncertainty.

### REFERENCES
